A retrospective on rigorous results on the Bose gas

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Foreword

Joel wanted a review of 50 years of statistical mechanics for the benefit of young people entering the field. Sort of like Bellamy's "looking backwards" but only a third as far. Like Bellamy, it will seem that what is known today looked impossibly difficult years ago. My message for the beginners is that progress does occur, and some of the difficult problems now being addressed in our field might actually get solved in your lifetime.

The Bose gas is a good topic because exactly 50 years ago I became a postdoc of Bethe and he asked me to look at the Lee-Huang-Yang 1957 paper. At that time, in order to understand the behavior of liquid helium, starting from Schrödinger's equation, theorists studied **the low density gas**. They still do.

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No one then ever thought they would see BEC for weakly interacting particles. The only system that was known was liquid helium. The modern discovery of BEC in cold gases was regarded, at first (by at least one very important physicist of my acquaintance), as a *"ho-hum so what's new"* result. It was hard, and it is still hard, to see all the repercussions of this experimental discovery. What we do know is that this ancient problem led us to tons of new physics.

Hamiltonian of an interacting gas

$$\boxed{H = -\mu \sum_{j=1}^{N} \nabla_j^2 \; + \sum_{1\le i<j\le N} v(x_i - x_j)} \qquad \text{with } \boxed{\mu = \hbar^2/2m \;.} \qquad (1)$$

- **Lenz**, (1929): For low density, ρ,

$$\boxed{\mathbf{E_0 = 4\pi\mu N\rho\, a}} \qquad \text{with } a = 2 \text{ body scattering length of } v. \qquad \textit{False in } 2D. \quad (2)$$

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• **Bogolubov** (1947) established the basic theory. He got it right, except for some problems of consistency. His formula showed that the ground state energy is

$$E_0 = \frac{1}{2} N\rho \int v(x)dx, \tag{5}$$

which is nonsense — but correct (if you see what I mean). He decided it should really be (4) because $\int v$ is the first Born approximation to $8\pi\mu a$

a depends on μ, which implies that *no naive perturbation theory is going to work!*

- **LHY** (1957) got the next term, (which was implicit in Bogolubov):

$$\boxed{E_0 = 4\pi\mu N\rho a\left(1 \ + \ \frac{\mathbf{128}}{\mathbf{15}\sqrt{\pi}}\sqrt{\rho\,\mathbf{a^3}}\right) \ .} \tag{6}$$

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What do these two terms represent? Why is it so difficult to derive them in a straightforward manner?

Answers: (1.) It is impossible to think of individual particles because the "uncertainty principle" length, $(\rho a)^{-1/2} >> \rho^{-1/3} =$ "average particle spacing". The first term is the energy needed to make an infinitesimally tiny hole in the fluid.

(2.) The tiny pebble of size a thrown into the gas produces a "splash" of size $(\rho a)^{-1/2} >> \rho^{-1/3}$ and energy of order $\rho a\sqrt{\rho a^3}$.

Moral; There are 3 length scales: $\boxed{\mathbf{a} << \rho^{-1/3} << (\rho\mathbf{a})^{-1/2}}$ and they are *connected!*. This is quantum mechanics at its best (worst?).

• **Dyson** (1957) had the crucial idea of how to transmute a hard core potential into a soft potential for a lower bound to the energy. Upper bounds are also very difficult to obtain, but he showed the way on this issue, too, and showed that

$$\boxed{E_0 < 4\pi\mu\rho a\left\{1 + C(\rho a^3)^{1/3}\right\}}$$

.

• **L, Yngvason** (1998), using ideas of Dyson, showed that (for $v \geq 0$)

$$\boxed{E_0 > 4\pi\mu\rho a\left\{1 - C\,(\rho a^3)^{1/17}\right\}} \tag{8}$$

This ends Chapter 1 (1947-1998), namely, understanding the first term in E_0.

What about the second Bogolubov term?

First answer : Let's look at a completely different problem – the *charged Bose gas.*

In this case there is no first term because of charge neutrality. Everything is correlation.

- **Foldy** (1961) calculated the energy of Jellium using Bogolubov's procedure.
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Second answer :

- **Giuliani, Seiringer**(2008). Let v scale as $v(x) \to \lambda^3 v(\lambda x)$,which preserves $\int v$. Take $\lambda = \rho^{\epsilon+1/3}$, with $\epsilon > 0$. Each particle overlaps with many others. Then the second Bogolubov energy is exact!

- **L, Solovej** (unpublished). Take $\lambda = \rho^{-\epsilon+1/3}$. Each particle overlaps with only one other. Then the second Bogolubov energy is exact!

Hope: Someday, maybe we can prove this with $\lambda = 1$.

What about Bose-Einstein condensation in the ground state?

Nobody is close to a proof for the thermodynamic limit .

Note that none of the foregoing proofs about E_0 need or imply condensation, even though Bogolubov used BEC in an important way in his calculation.

What has been proved so far is a partial result:

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- **L, Seiringer** (2002). There is *one hundred percent* BE condensation (and superfluidity, too,) in traps in the Gross-Pitaevski limit $N \to \infty \quad a \to 0$ with both Na and the trap potential V fixed.

Gross-Pitaevskii equation

In a trapping potential $V(x)$ for the low density gas, the density $\rho(x) = |\Phi(x)|^2$ satisfies

$$\boxed{\left\{-\mu\nabla^2 + V(x) + 8\pi\mu a|\Phi(x)|^2\right\}\Phi(x) = \gamma\Phi(x).} \tag{9}$$

- **L, Seiringer, Yngvason** (2000) proved this in the **GP limit:**
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- **L, Seiringer** (2005) The GP equation also holds for a **rapidly rotating** gas. This is not obvious since the Bose ground state might not be the absolute ground state. In rotating coordinates, with rotation frequency $\vec{\Omega}$,

$$\left\{-\mu\nabla^2 - \tilde{\mathbf{\Omega}}\cdot\tilde{\mathbf{L}} + V(x) + 8\pi\mu a|\Phi(x)|^2\right\}\Phi(x) = \gamma\Phi(x). \tag{11}$$

  An outstanding, recent development is

- **Erdös, Schlein, Yau** For suitable initial many-body data, the **time-dependent** GP equation is correct: $\boxed{\left\{-\mu\nabla^2 + V(x) + 8\pi\mu a|\Phi(x)|^2\right\}\Phi(x) = i\dot{\mathbf{\Phi}}(x).}$

Other things I could mention, but don't have time for

One-D

Two-D

Optical lattices

☺ Looking forward to the 200^{th}! ☺