Using sharp-threshold theorems in statistical mechanics

Geoffrey Grimmett

SMM100
17 December 2008

Influence

$X_1, X_2, \ldots, X_n$ independent coin tosses, density p
Configurations ω lie in the *sample space* (poset) $\Omega = \{0,1\}^n$
Event $A \subseteq \Omega$ is *increasing* if: $\omega \in A,\ \omega \le \omega' \Rightarrow \omega' \in A$
$\omega_i \equiv \omega$ with $\omega(i) = 1$, and $\omega^i \equiv \omega$ with $\omega(i) = 1$

Defn: The (absolute) influence of coin i on event A is

$$I_A(i) = \mu_p\big(1_A(\omega_i) \ne 1_A(\omega^i)\big).$$

- Voting:
  $I_A(i) = \mu_p$(voter i can influence the occurrence of event A)
- Pivotality: If A is increasing,

$$\begin{aligned} I_A(i) &= \mu_p(A \mid \omega(i) = 1) - \mu_p(A \mid \omega(i) = 0) \\ &= \mu_p(i \text{ pivotal for } A) \end{aligned}$$

Influence

$X_1, X_2, \dots, X_n$ independent coin tosses, density p
Configurations ω lie in the *sample space* (poset) $\Omega = \{0,1\}^n$
Event $A \subseteq \Omega$ is *increasing* if: $\omega \in A,\ \omega \le \omega' \Rightarrow \omega' \in A$
$\omega_i \equiv \omega$ with $\omega(i) = 1$, and $\omega^i \equiv \omega$ with $\omega(i) = 1$

Defn: The (absolute) influence of coin i on event A is

$$I_A(i) = \mu_p\big(1_A(\omega_i) \ne 1_A(\omega^i)\big).$$

- Voting:
  $I_A(i) = \mu_p\big(\text{voter } i \text{ can influence the occurrence of event } A\big)$
- Pivotality: If A is increasing,

$$\begin{aligned} I_A(i) &= \mu_p(A \mid \omega(i) = 1) - \mu_p(A \mid \omega(i) = 0) \\ &= \mu_p(i \text{ pivotal for } A) \end{aligned}$$

Main theorem for coin tosses

Theorem (Kahn–Kalai–Linial, Talagrand)

$$\sum_i I_A(i) \geq c\mu_p(A)\mu_p(\overline{A}) \log \left\{ \frac{1}{\max_i I_A(i)} \right\}$$

Corollary: $M = \max_i I_A(i)$ satisfies $nM \geq \cdots \log(1/M)$, so

$$M \geq c' \mu_p(a)\mu_p(\overline{A}) \frac{\log n}{n}.$$

Optimality of $n^{-1} \log n$: Israeli tribes example

Main theorem for coin tosses

Theorem (Kahn–Kalai–Linial, Talagrand)

$$\sum_i I_A(i) \geq c\mu_p(A)\mu_p(\overline{A}) \log \left\{ \frac{1}{\max_i I_A(i)} \right\}$$

Corollary: $M = \max_i I_A(i)$ satisfies $nM \geq \cdots \log(1/M)$, so

$$M \geq c'\mu_p(a)\mu_p(\overline{A}) \frac{\log n}{n}.$$

Optimality of $n^{-1} \log n$: Israeli tribes example

Russo–Margulis $+\cdots$: For increasing A

$$\frac{d}{dp}\mu_p(A) = \sum_i I_A(i)$$

Theorem (Sharp threshold)

$$\frac{d}{dp}\mu_p(A) \geq c\mu_p(A)\mu_p(\overline{A}) \log[1/M]$$

where $M = M_p = \max_i I_A(i)$

Corollary: If $M_p \leq K$, then $\mu_p(A)$ passes from ϵ to $1-\epsilon$ on an interval of length $\leq C/\log[1/K]$

Find upper bound for M_p!

Russo–Margulis $+\cdots$: For increasing A

$$\frac{d}{dp}\mu_p(A) = \sum_i I_A(i)$$

Theorem (Sharp threshold)

$$\frac{d}{dp}\mu_p(A) \geq c\mu_p(A)\mu_p(\overline{A}) \log[1/M]$$

where $M = M_p = \max_i I_A(i)$

Corollary: If $M_p \leq K$, then $\mu_p(A)$ passes from ϵ to $1-\epsilon$ on an interval of length $\leq C/\log[1/K]$

Find upper bound for M_p!

Proof of KKL-T when $p = \frac{1}{2}$

Fourier space Ω

Orthonormal basis $u_F(\omega) = \prod_{i \in F} (-1)^{\omega(i)}$

$C = \{f : \Omega \to \mathbb{R}\}$, Inner product $\langle fg \rangle = \mu_{\frac{1}{2}}(fg)$

Fourier representation $f = \sum_F \widehat{f}(F) u_F$

Influence: $I_A(i) = 4 \sum_{F \ni i} \widehat{1}_A(F)^2, \quad \sum_i I_A(i) = 4 \sum_F |F| \widehat{1}_A(F)^2$

Hypercontractivity: $T_\rho g = \sum_F \rho^{|F|} \widehat{g}(F) u_F = E(g(\Psi))$

where Ψ is obtained by re-sampling ω with local density $1 - \rho$.

"Noise sensitivity"

Proof of KKL-T when $p = \frac{1}{2}$

Fourier space Ω

Orthonormal basis $u_F(\omega) = \prod_{i \in F}(-1)^{\omega(i)}$

$C = \{f : \Omega \to \mathbb{R}\}$, Inner product $\langle fg \rangle = \mu_{\frac{1}{2}}(fg)$

Fourier representation $f = \sum_F \widehat{f}(F) u_F$

Influence: $I_A(i) = 4 \sum_{F \ni i} \widehat{1}_A(F)^2, \quad \sum_i I_A(i) = 4 \sum_F |F| \widehat{1}_A(F)^2$

Hypercontractivity: $T_\rho g = \sum_F \rho^{|F|} \widehat{g}(F) u_F = E(g(\Psi))$

where Ψ is obtained by re-sampling ω with local density $1 - \rho$.

"Noise sensitivity"

Dynamic(al) percolation

Critical site percolation on triangular lattice, $p = \frac{1}{2}$.

Refreshment of local states at rate 1

No percolation at $p_c = \frac{1}{2}$

Theorem (Garban–Pete–Schramm, arxiv:0803.3750)

Let T be the set of times when there exists an infinite black cluster. Almost surely:

$$\dim(T) = \frac{31}{36}, \quad \dim(T_{\mathbb{Z}\times\mathbb{Z}_+}) = \frac{5}{9},$$
$$\dim(T(\text{black and white})) \geq \frac{1}{9}.$$

Method: by estimating spectra

Monotone measures

Defn: The (positive) probability measure μ on Ω is *monotone* if:

$$\mu(X_i = 1 \mid X_j = \xi_j \text{ for } j \neq i) \quad \text{is increasing in } \xi$$

The (conditional) influence of variable i on event A is

$$J_A(i) = \mu(A \mid \omega(i) = 1) - \mu(A \mid \omega(i) = 0)$$

FKG/Holley: μ is monotone iff it satisfies the Holley condition

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2)$$

where $\omega_1 \vee \omega_2$ denotes pointwise maximum, and $\wedge$ pointwise minimum.

- product measure (independence), $\mu = \mu_{\frac{1}{2}}$
- random-cluster measure, $\mu(\omega) \propto q^{k(\omega)}$
- Ising measure, $\mu(\sigma) \propto e^{-\beta|+-|}$

Monotone measures

Defn: The (positive) probability measure μ on Ω is *monotone* if:

$$\mu(X_i = 1 \mid X_j = \xi_j \text{ for } j \neq i) \quad \text{is increasing in } \xi$$

The (conditional) influence of variable i on event A is

$$J_A(i) = \mu(A \mid \omega(i) = 1) - \mu(A \mid \omega(i) = 0)$$

FKG/Holley: μ is monotone iff it satisfies the Holley condition

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2)$$

where $\omega_1 \vee \omega_2$ denotes pointwise maximum, and $\wedge$ pointwise minimum.

- product measure (independence), $\mu = \mu_{\frac{1}{2}}$
- random-cluster measure, $\mu(\omega) \propto q^{k(\omega)}$
- Ising measure, $\mu(\sigma) \propto e^{-\beta|+-|}$

Monotone with 'external field' μ monotone, $p \in [0,1]$

$$\mu_p(\omega) = \frac{1}{Z}\mu(\omega)\prod_i \left\{p^{\omega(i)}(1-p)^{1-\omega(i)}\right\}$$
$$= \frac{1}{Z}p^{|\omega|}(1-p)^{n-|\omega|}\mu(\omega), \qquad |\omega| := \sum_i \omega(i)$$

Examples:

- product measure, μ_p
- random-cluster measure, $\phi_{p,q}(\omega) \propto p^{|\omega|}(1-p)^{n-|\omega|}q^{k(\omega)}$
- Ising with external field, $\mu_h(\sigma) \propto e^{h|\sigma|}e^{-\beta|+-|}$

Theorem (Graham–G)

$$\sum_i J_A(i) \geq c\mu_p(A)\mu_p(\overline{A}) \log\left\{\frac{1}{2\max_i J_A(i)}\right\}$$

BGK: For increasing A

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)}\mathrm{cov}_p(1_A, |\omega|)$$

Theorem (Sharp threshold)

$$\frac{d}{dp}\mu_p(A) \geq \frac{c\xi_p}{p(1-p)}\mu_p(A)\mu_p(\overline{A}) \log[1/(2M)]$$

where $M = M_p = \max_i J_A(i)$ *and* $\xi_p = \min_i\{\mu_p(X_i)\mu_p(1 - X_i)\}$

Theorem (Graham–G)

$$\sum_i J_A(i) \geq c\mu_p(A)\mu_p(\overline{A}) \log \left\{ \frac{1}{2 \max_i J_A(i)} \right\}$$

BGK: For increasing A

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)} \mathrm{cov}_p(1_A, |\omega|)$$

Theorem (Sharp threshold)

$$\frac{d}{dp}\mu_p(A) \geq \frac{c\xi_p}{p(1-p)} \mu_p(A)\mu_p(\overline{A}) \log[1/(2M)]$$

where $M = M_p = \max_i J_A(i)$ *and* $\xi_p = \min_i\{\mu_p(X_i)\mu_p(1 - X_i)\}$

Further extensions

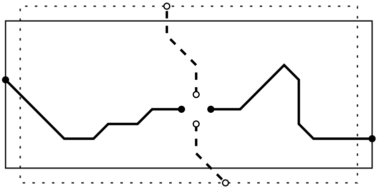
Similar influence theorems hold for:

- [BKKKL] a family of Uniform$[0, 1]$ random variables
- [BKKKL, G] a family of n iid random variables on a probability space satisfying: the associated measure ring of the non-atomic part is separable
- a family of Bernoulli (p) variables
- monotone and non-monotone events

Notes (on this and more):
`www.statslab.cam.ac.uk/~grg/books/pgs.html`

Percolation on $\mathbb{Z}^2$

Bond percolation on $\mathbb{Z}^2$, box $B_n = [0, n+1] \times [0, n]$



$A = \{\text{left–right crossing of } B_n\}$

$$I_A(e) = \mu_p(e \text{ pivotal}) \\ \leq \mu_{\frac{1}{2}}(0 \leftrightarrow \partial B_{n/2}) \to 0 \quad \text{as } n \to \infty$$

By sharp-threshold theorem,

$$\frac{d}{dp}\mu_p(A) \geq c\mu_p(A)\mu_p(\overline{A}) \log\left\{\frac{1}{\mu_{\frac{1}{2}}(0 \leftrightarrow \partial B_{n/2})}\right\}$$

Duality: $\mu_{\frac{1}{2}}(A) = \frac{1}{2}$, therefore, for $p > \frac{1}{2}$, $\mu_p(A) \to 1$ as $n \to \infty$
Variety of consequences using box-crossing arguments (RSW):
$p_c = \frac{1}{2}$, exponential decay, etc

K BR S Z

Random-cluster model on $\mathbb{Z}^2$

Graph $G = (V, E)$, edges are open/closed

$$\phi_{p,q}(\omega) = \frac{1}{Z} p^{|\omega|}(1-p)^{|E|-|\omega|} q^{k(\omega)}, \quad \omega \in \Omega = \{0,1\}^E$$

$|\omega|$ = number of open edges, $k(\omega)$ = number of open clusters

Self-dual point on $\mathbb{Z}^2$: $p_{\mathrm{sd}}(q) = \dfrac{\sqrt{q}}{1+\sqrt{q}}$

Conjecture

$p_{\mathrm{c}}(q) = p_{\mathrm{sd}}(q)$ for $q \geq 1$.

Known: $q = 1$ (Kesten), $q = 2$ ('Onsager'), $q \geq 25.72$

Random-cluster model on $\mathbb{Z}^2$

Graph $G = (V, E)$, edges are open/closed

$$\phi_{p,q}(\omega) = \frac{1}{Z} p^{|\omega|}(1-p)^{|E|-|\omega|} q^{k(\omega)}, \quad \omega \in \Omega = \{0,1\}^E$$

$|\omega|$ = number of open edges, $\quad k(\omega)$ = number of open clusters

Self-dual point on $\mathbb{Z}^2$: $p_{\mathrm{sd}}(q) = \dfrac{\sqrt{q}}{1+\sqrt{q}}$

Conjecture

$p_{\mathrm{c}}(q) = p_{\mathrm{sd}}(q)$ *for* $q \geq 1$.

Known: $q = 1$ (Kesten), $q = 2$ ('Onsager'), $q \geq 25.72$

Theorem (G–G)

Let $q \geq 1$. The probability of a box-crossing of B_n increases steeply from ~ 0 to ~ 1 as p passes through $p_{\mathrm{sd}}(q)$.

Proof: Use coupling to bound the conditional influences

$$\begin{aligned} J_A(e) &= \phi_{p,q}(A \mid e \text{ open}) - \phi_{p,q}(A \mid e \text{ closed}) \\ &\leq \phi_{p,q}\big(B_n \text{ crossed within } C_e \mid e \text{ open}\big) \end{aligned}$$

where C_e is the open cluster at e.

Input: Absence of percolation when $p = p_{\mathrm{sd}}(q)$, (Zhang)

Theorem (G–G)

Let $q \geq 1$. The probability of a box-crossing of B_n increases steeply from ~ 0 to ~ 1 as p passes through $p_{\mathrm{sd}}(q)$.

Proof: Use coupling to bound the conditional influences

$$\begin{aligned} J_A(e) &= \phi_{p,q}(A \mid e \text{ open}) - \phi_{p,q}(A \mid e \text{ closed}) \\ &\leq \phi_{p,q}\big(B_n \text{ crossed within } C_e \mid e \text{ open}\big) \end{aligned}$$

where C_e is the open cluster at e.

Input: Absence of percolation when $p = p_{\mathrm{sd}}(q)$, (Zhang)

What's missing?

Answer: RSW for RCM.

Given square-crossings, how to build rectangle-crossings?

FKG, but no estimate of correlation-decay

Ising model with external field

Ising model on $\mathbb{Z}^2$, inverse-temperature β, external field h

Qn: When are there infinite + clusters?

Answer: (Higuchi) Iff $h > h_c(\beta)$ where

$$h_c(\beta) \begin{cases} > 0 & \text{if } \beta > \beta_c, \\ = 0 & \text{if } \beta < \beta_c. \end{cases}$$

Method: Sharp-threshold plus RSW (using exponential decay of correlations)

Ising model with external field

Ising model on $\mathbb{Z}^2$, inverse-temperature β, external field h

Qn: When are there infinite $+$ clusters?

Answer: (Higuchi) Iff $h > h_c(\beta)$ where

$$h_c(\beta) \begin{cases} > 0 & \text{if } \beta > \beta_c, \\ = 0 & \text{if } \beta < \beta_c. \end{cases}$$

Method: Sharp-threshold plus RSW (using exponential decay of correlations)

Sharp-threshold theorem applies, as for RCM.

Input: No infinite + cluster when $h = h_c(\beta)$, (Zhang)

vdB

Note: Simple proof (Werner) of continuity of magnetization in $h = 0$ Ising model at $\beta = \beta_c$.

Sharp-threshold theorem applies, as for RCM.

Input: No infinite + cluster when $h = h_c(\beta)$, (Zhang)

vdB

Note: Simple proof (Werner) of continuity of magnetization in $h = 0$ Ising model at $\beta = \beta_c$.

Box-crossings in other systems

I. Coloured random-cluster model

Sample from $\phi_{p,q}$

Colour each cluster *black* $(+1)$ with probability α, otherwise *white* (-1)

Spin-measure $\mu_{p,q,\alpha}$, with $q\alpha, q(1-\alpha) \geq 1$

Measure: Add external field on black vertices

$$\mu_h(\sigma) \propto e^{h|\sigma|}\mu_{p,q,\alpha}(\sigma), \qquad |\sigma| := \#\{\text{black sites}\}$$

Look at black crossings of large boxes

II. Massively coloured random-cluster model

Condition $\phi_{p,q} \times \mu_\alpha$ on $\{$colours are constant on clusters$\}$

$$\psi_{p,q,\alpha}(\sigma) \propto \left(\frac{\alpha}{1-\alpha}\right)^{|\sigma|} (1-p)^{|+-|} Z_{p,q,+} Z_{p,q,-}$$

Both I and II contain Ising model when $\alpha = \frac{1}{2}$

Finally . . .

Influence and sharp-threshold:

- a beautiful theory with the capacity to solve problems
- a robust method for proving steepness
- a method in search of applications

References

- (Graham–G) Annals of Probability 34 (2006) 2006.
- (Graham–G) Sharp thresholds for the random-cluster and Ising models, preprint, 2008
- *Probability on Graphs*, www.statslab.cam.ac.uk/∼grg/books/pgs.html