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Short-Range Spin Glasses

Looking back, looking forward

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Consider:
configurations of Ising spins

$$\sigma = \{\sigma_i\}, \quad \tau = \{\tau_i\}, \quad \ldots$$

Define:
the standard overlap among them

$$q_N(\sigma,\tau) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i$$

or, if the spins are sitting in a box Λ of a d-dimensional square lattice, the link-overlap

$$Q_\Lambda(\sigma,\tau) = \frac{1}{d|\Lambda|}\sum_{|i-j|=1}\sigma_i\sigma_j\tau_i\tau_j$$

Introduce:
for $\beta > 0$ the random probability measure

$$p_\Lambda(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{\sum_\sigma e^{-\beta H_\Lambda(\sigma)}}$$

with $\{H_\Lambda(\sigma)\}$ a centered Gaussian family defined by the covariance

$$\mathrm{Av}(H_\Lambda(\sigma)H_\Lambda(\tau)) = |\Lambda|Q_\Lambda(\sigma,\tau)$$

...that defines the Edwards-Anderson model

$$H_\Lambda(\sigma) \;=\; -\sum_{|i-j|=1} J_{i,j}\sigma_i\sigma_j$$

Quantities of interest:

- pressure

$$P_\Lambda(\beta) \;=\; \mathsf{Av}\log\sum_\sigma e^{-\beta H_\Lambda(\sigma)}$$

- moments

$$\mathsf{Av}\left(\frac{\sum_{\sigma,\tau} Q_\Lambda(\sigma,\tau)e^{-\beta[H_\Lambda(\sigma)+H_\Lambda(\tau)]}}{\sum_{\sigma,\tau} e^{-\beta[H_\Lambda(\sigma)+H_\Lambda(\tau)]}}\right) \;=$$

$$:=< Q_{12} >_\Lambda := \int Q p_\Lambda(Q) dQ$$

$$< Q_{12}Q_{23} >_\Lambda = \int Q_{12}Q_{23}p_\Lambda^{(12),(23)}(Q_{12},Q_{23})$$

$< Q_{12}Q_{34} >_\Lambda$, $< Q_{1,2}Q_{2,3}Q_{3,1} >_\Lambda$ etc.

What do we know?

Inequality:

$$\mathrm{Av}(J_X \omega_X) \geq 0$$

P.C., Sandro Graffi, Gaussian case, CMP 2004
P.C., Joel Lebowitz, general case, AHP 2007

- sub-additivity of the pressure, thermodynamical limit

$$p = \lim_{\Lambda \nearrow \mathbf{Z}^d} p_\Lambda \,, \quad p_\Lambda = \frac{1}{|\Lambda|} P_\Lambda$$

- upper and lower bounds for the surface pressure:

$$\tau_\Lambda = \frac{1}{|\partial\Lambda|}[P_\Lambda - p|\Lambda|]$$

- τ depends on b.c.

How close (or far) is from mean field?

The Parisi Solution of the Sherrington-Kirkpatrick model is based on two assumptions

- Replica Equivalence

$$p^{(12),(23)}(q_{12},q_{23}) = \frac{1}{2}p(q_{12})\delta(q_{12}-q_{23})+\frac{1}{2}p(q_{12})p(q_{23})$$

$$p^{(12),(34)}(q_{12},q_{34}) = \frac{1}{3}p(q_{12})\delta(q_{12}-q_{34})+\frac{2}{3}p(q_{12})p(q_{34})$$

- Ultrametricity

$$p^{(12),(23),(31)}(q_{12},q_{23},q_{31}) =$$

$$\delta(q_{12}-q_{23})\delta(q_{23}-q_{31})p(q_{12})\int_0^{q_{12}} dq p(q)$$

$$+\,\theta(q_{12}-q_{23})\delta(q_{23}-q_{31})p(q_{12})p(q_{23})$$

$$+\,2 \text{ cyclic permutations}$$

- Replica Equivalence and Ultrametricity allows to reconstruct the $p_N^{(\{kl\})}(\{q_{kl}\})$ starting from $p(q)$!

Replica Equivalence:

1995 F.Guerra,
1997 M.Aizenman and P.C.,
1998 S.Ghirlanda and F.Guerra

Questions:

- What happens in finite dimensional models (in Z^d)?

- Does the same factorization structure apply?

- Replica equivalence? Ultrametricity?

Result:

- Edwards-Anderson fulfills Replica Equivalence in terms of the link-overlap.

P.C., Cristian Giardinà, AHP 2005, JSP 2007

Remarks:
- mean field feature
- compatible with different low temperature pictures

Theorem:
Let h be the Hamiltonian per particle. For every smooth bounded function $\mathcal{O}$ and for all intervals β_1, β_2

$$\int_{\beta_1}^{\beta_2} d\beta (< h\mathcal{O} > - < h >< \mathcal{O} >) \to 0$$

The Replica Equivalence identities follow integrating by parts.

Proof ideas:

- control of energy fluctuation in the thermodynamical limit

- energy per particle tends a.e. to a constant with respect to the equilibrium measure for large volumes (a part of isolated singularities)

$$< h^2 > - < h >^2 \rightarrow 0$$

Proved by Stochastic Stability,

$$\int_{\beta_1}^{\beta_2} d\beta \, \mathsf{Av}[\Omega(h^2)] - \mathsf{Av}[\Omega(h)^2] \rightarrow 0$$

and Self Averaging:

$$\int_{\beta_1}^{\beta_2} d\beta \, \mathsf{Av}[\Omega(h)^2] - \mathsf{Av}[\Omega(h)]^2 \rightarrow 0$$

Interaction Flip Identities.

P.C., Cristian Giardina, Claudio Giberti (arXiv:0811.2472)

Symmetry of the distribution:

$$J_{i,j} \rightarrow -J_{i,j}$$

Stationary Interpolation

$$H_\Lambda(\sigma)_t \ = \ \cos(t) H_\Lambda(\sigma) + \sin(t) \tilde{H}_\Lambda(\sigma)$$

$$\mathcal{P}_\Lambda(t) \ = \ \log \sum_\sigma e^{-H_\Lambda(\sigma)_t}$$

$$\mathcal{X}_\Lambda(a,b) \ = \ \mathcal{P}_\Lambda(b) - \mathcal{P}_\Lambda(a)$$

$$\text{Av}\mathcal{X}_\Lambda(a,b) \;=\; 0\;.$$

Bound on fluctuations:

$$\text{Av}\mathcal{X}_\Lambda^2(a,b) \;\leq\; c|\Lambda|$$

Theorem:

The (s,t) integral on all $[a,b]^2$ with the positive kernel $\sin^2(t-s)$ of the quantity

$$< C_{1,2}^2 >_{t,s} -2 < C_{1,2}C_{2,3} >_{s,t,s} + < C_{1,2}C_{3,4} >_{t,s,s,t}$$

vanishes for large volumes.

NUMERICAL RESULTS, parallel tempering, up to linear sizes $L = 20$, for both Gaussian or Bernoulli distribution.

1) Overlap Equivalence:
(P.C., C.Giardinà, C.Giberti, C.Vernia, PRL 2006)

Consider q and Q in $d = 3$

- They are uncorrelated at high temperatures ($T > T_c$)

- Below T_C, $Var_L(Q|q^2)$ scales like $L^{-\alpha}$

- Q is a monotonic increasing function of q^2: $< Q|q^2 >$

Contradicts the T.N.T. picture: non-trivial standard overlap distribution, trivial link-overlap distribution.

2) Ultrametricity:
(P.C., C.Giardinà, C.Giberti, G.Parisi, C.Vernia, PRL 2007)

Consider the overlap triangle of sides $Q_{1,2}$, $Q_{2,3}$, $Q_{3,1}$, introduce the random variables

$$Q_{max} = max(Q_{1,2}, Q_{2,3}, Q_{3,1})$$

$$Q_{med} = med(Q_{1,2}, Q_{2,3}, Q_{3,1})$$

$$Q_{min} = min(Q_{1,2}, Q_{2,3}, Q_{3,1})$$

study the distributions of

$$X = Q_{med} - Q_{min}$$

$$Y = Q_{max} - Q_{med}$$

- Variance of X converges to zero by a scaling law of $N^{-\gamma}$ (no scalene triangles!)

- Variance of Y doesn't (isosceles triangles!)