Short-Range Spin Glasses

Looking back, looking forward

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Consider:
configurations of Ising spins

\[ \sigma = \{ \sigma_i \}, \quad \tau = \{ \tau_i \}, \quad \ldots \]

Define:
the standard overlap among them

\[ q_N(\sigma, \tau) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tau_i \]

or, if the spins are sitting in a box \( \Lambda \) of a d-dimensional square lattice, the link-overlap

\[ Q_\Lambda(\sigma, \tau) = \frac{1}{d|\Lambda|} \sum_{|i-j|=1} \sigma_i \sigma_j \tau_i \tau_j \]

Introduce:
for \( \beta > 0 \) the random probability measure

\[ p_\Lambda(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{\sum_{\sigma} e^{-\beta H_\Lambda(\sigma)}} \]

with \( \{ H_\Lambda(\sigma) \} \) a centered Gaussian family defined by the covariance

\[ \text{Av}(H_\Lambda(\sigma)H_\Lambda(\tau)) = |\Lambda|Q_\Lambda(\sigma, \tau) \]
...that defines the Edwards-Anderson model

\[ H_\Lambda(\sigma) = - \sum_{|i-j|=1} J_{i,j} \sigma_i \sigma_j \]

Quantities of interest:

- **pressure**

\[ P_\Lambda(\beta) = \text{Av} \log \sum_{\sigma} e^{-\beta H_\Lambda(\sigma)} \]

- **moments**

\[
\text{Av} \left( \frac{\sum_{\sigma,\tau} Q_\Lambda(\sigma, \tau) e^{-\beta[H_\Lambda(\sigma) + H_\Lambda(\tau)]}}{\sum_{\sigma,\tau} e^{-\beta[H_\Lambda(\sigma) + H_\Lambda(\tau)]}} \right) = \]

\[ := \langle Q_{12} \rangle_\Lambda := \int Q_{p_\Lambda}(Q) dQ \]

\[ \langle Q_{12} Q_{23} \rangle_\Lambda = \int Q_{12} Q_{23} p_{\Lambda}^{(12),(23)}(Q_{12}, Q_{23}) \]

\[ \langle Q_{12} Q_{34} \rangle_\Lambda, \langle Q_{1,2} Q_{2,3} Q_{3,1} \rangle_\Lambda \text{ etc.} \]
What do we know?

Inequality:

\[ \text{Av}(J_X \omega_X) \geq 0 \]

P.C., Sandro Graffi, Gaussian case, CMP 2004
P.C., Joel Lebowitz, general case, AHP 2007

- sub-additivity of the pressure, thermodynamical limit

\[ p = \lim_{\Lambda \to \mathbb{Z}^d} p_\Lambda, \quad p_\Lambda = \frac{1}{|\Lambda|} P_\Lambda \]

- upper and lower bounds for the surface pressure:

\[ \tau_\Lambda = \frac{1}{|\partial \Lambda|} [P_\Lambda - p|\Lambda|] \]

- \( \tau \) depends on b.c.
How close (or far) is from mean field?

The Parisi Solution of the Sherrington-Kirkpatrick model is based on two assumptions

- Replica Equivalence

\[ p^{(12),(23)}(q_{12}, q_{23}) = \frac{1}{2} p(q_{12}) \delta(q_{12} - q_{23}) + \frac{1}{2} p(q_{12}) p(q_{23}) \]

\[ p^{(12),(34)}(q_{12}, q_{34}) = \frac{1}{3} p(q_{12}) \delta(q_{12} - q_{34}) + \frac{2}{3} p(q_{12}) p(q_{34}) \]

- Ultrametricity

\[ p^{(12),(23),(31)}(q_{12}, q_{23}, q_{31}) = \]

\[ \delta(q_{12} - q_{23}) \delta(q_{23} - q_{31}) p(q_{12}) \int_{0}^{q_{12}} dq p(q) \]

\[ + \theta(q_{12} - q_{23}) \delta(q_{23} - q_{31}) p(q_{12}) p(q_{23}) \]

\[ + 2 \text{ cyclic permutations} \]
• Replica Equivalence and Ultrametricity allows to reconstruct the $p_N^{\{kl\}}(\{q_{kl}\})$ starting from $p(q)$!

Replica Equivalence:

1995 F.Guerra,
1997 M.Aizenman and P.C.,
1998 S.Ghiroldana and F.Guerra

Questions:

• What happens in finite dimensional models (in $Z^d$)?

• Does the same factorization structure apply?

• Replica equivalence? Ultrametricity?
**Result:**

- Edwards-Anderson fulfills Replica Equivalence in terms of the link-overlap.

P.C., Cristian Giardinà, AHP 2005, JSP 2007

**Remarks:**
- mean field feature
- compatible with different low temperature pictures

**Theorem:**
Let $h$ be the Hamiltonian per particle. For every smooth bounded function $\mathcal{O}$ and for all intervals $\beta_1, \beta_2$

$$
\int_{\beta_1}^{\beta_2} d\beta (\langle h \mathcal{O} \rangle - \langle h \rangle \langle \mathcal{O} \rangle) \rightarrow 0
$$

The Replica Equivalence identities follow integrating by parts.
**Proof ideas:**

- control of energy fluctuation in the thermodynamical limit

- energy per particle tends a.e. to a constant with respect to the equilibrium measure for large volumes (a part of isolated singularities)

\[
<h^2> - <h>^2 \to 0
\]

Proved by Stochastic Stability,

\[
\int_{\beta_1}^{\beta_2} d\beta \, \text{Av}[\Omega(h^2)] - \text{Av}[\Omega(h)^2] \to 0
\]

and Self Averaging:

\[
\int_{\beta_1}^{\beta_2} d\beta \, \text{Av}[\Omega(h)^2] - \text{Av}[\Omega(h)]^2 \to 0
\]
Interaction Flip Identities.

P.C., Cristian Giardina, Claudio Giberti (arXiv:0811.2472)

Symmetry of the distribution:

\[ J_{i,j} \rightarrow -J_{i,j} \]

Stationary Interpolation

\[
H_{\Lambda}(\sigma)_t = \cos(t) H_{\Lambda}(\sigma) + \sin(t) \tilde{H}_{\Lambda}(\sigma)
\]

\[
P_{\Lambda}(t) = \log \sum_{\sigma} e^{-H_{\Lambda}(\sigma)_t}
\]

\[
\mathcal{X}_{\Lambda}(a, b) = P_{\Lambda}(b) - P_{\Lambda}(a)
\]
\[ \text{Av} \mathcal{X}_\Lambda(a, b) = 0. \]

Bound on fluctuations:

\[ \text{Av} \mathcal{X}_\Lambda^2(a, b) \leq c|\Lambda| \]

**Theorem:**

The \((s, t)\) integral on all \([a, b]^2\) with the positive kernel \(\sin^2(t - s)\) of the quantity

\[ < C_{1,2}^2 >_{t,s} - 2 < C_{1,2} C_{2,3} >_{s,t,s} + < C_{1,2} C_{3,4} >_{t,s,s,t} \]

vanishes for large volumes.
NUMERICAL RESULTS, parallel tempering, up to linear sizes \( L = 20 \), for both Gaussian or Bernoulli distribution.

1) **Overlap Equivalence:**
(P.C., C.Giardinà, C.Giberti, C.Vernia, PRL 2006)

Consider \( q \) and \( Q \) in \( d = 3 \)

- They are uncorrelated at high temperatures \( (T > T_c) \)

- Below \( T_C \), \( \text{Var}_L(Q|q^2) \) scales like \( L^{-\alpha} \)

- \( Q \) is a monotonic increasing function of \( q^2 \):
  \[ < Q|q^2 > \]

Contradicts the T.N.T. picture: non-trivial standard overlap distribution, trivial link-overlap distribution.
2) *Ultrametricity*:
(P.C., C.Giardinà, C.Giberti, G.Parisi, C.Vernia, PRL 2007)

Consider the overlap triangle of sides $Q_{1,2}$, $Q_{2,3}$, $Q_{3,1}$, introduce the random variables

$$Q_{\text{max}} = \max(Q_{1,2}, Q_{2,3}, Q_{3,1})$$
$$Q_{\text{med}} = \text{med}(Q_{1,2}, Q_{2,3}, Q_{3,1})$$
$$Q_{\text{min}} = \min(Q_{1,2}, Q_{2,3}, Q_{3,1})$$

study the distributions of

$$X = Q_{\text{med}} - Q_{\text{min}}$$
$$Y = Q_{\text{max}} - Q_{\text{med}}$$

- Variance of $X$ converges to zero by a scaling law of $N^{-\gamma}$ (no scalene triangles!)

- Variance of $Y$ doesn’t (isosceles triangles!)