Universality, Integrability and Analyticity in Critical Phenomena

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Analyticity I: Landau Theory

- Landau free energy

$$F[\mathbf{M}] = \int \left((\nabla \mathbf{M})^2 + \mathrm{Tr}(\mathbf{H} \cdot \mathbf{M}) + r_0 \mathrm{Tr} \mathbf{M}^2 + \lambda_3 \mathrm{Tr} \mathbf{M}^3 + \cdots \right) d^d x$$

- all analytic terms allowed by symmetry
- $\delta F / \delta \mathbf{M} = 0 \Rightarrow$ *critical points* at special values of $\{\lambda_j\} = (\mathbf{H}, r_0, \lambda_3, \ldots)$
- singular behavior from *bifurcations*
- *critical exponents* describing how $\mathbf{M}$ behaves close to critical points are *super-universal*.

Analyticity II: Renormalization Group

$$Z[\mathbf{H}] = \int_{|\mathbf{q}|<\Lambda} [d\mathbf{M}(x)]\, e^{-F[\mathbf{M}]}$$

$$\frac{d\lambda_j}{d\ell} = -\Lambda \frac{\partial \lambda_j}{\partial \Lambda} = -\beta_j(\{\lambda\})$$

- *fixed points* $\beta_j(\{\lambda^*\}) = 0$
- $\beta_j(\{\lambda\})$ assumed to be analytic in neighborhood
- singular behavior from infinite iterations of an analytic mapping
- critical exponents and other universal critical properties given by derivatives of β-functions
- scaling fields $\langle \phi_j(r_1)\phi_j(r_2)\rangle \sim |r_1 - r_2|^{-2x_j}$

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- scaling fields $\langle \phi_j(r_1)\phi_j(r_2)\rangle \sim |r_1 - r_2|^{-2x_j}$
- the only evidence we have for this picture being precisely correct for non-trivial cases is
  - perturbative analysis about a trivial fixed point (*e.g.* ϵ-expansion)
  - integrable lattice models in $d = 2$

Universality I: Renormalization Group

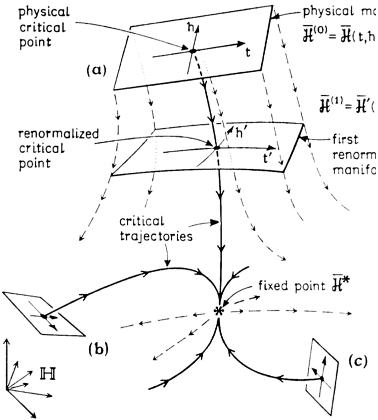
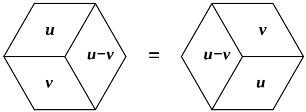


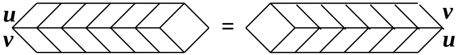
Figure: M E Fisher, Rev. Mod. Phys. **70** 653 (1998) [and `http://terpconnect.umd.edu/~xpectnil/`]

Analyticity III: Conformal Field Theory

- at (isotropic) 2d RG fixed points there are special scaling fields whose correlation functions are *holomorphic* (analytic) functions of $z = x + iy$ (or $\bar{z} = x - iy$):
- conserved currents corresponding to symmetries, *e.g.* stress tensor $\big(T(z), \overline{T}(\bar{z})\big)$
- *parafermionic fields* $\psi_\sigma(z)$ with $\langle \psi_\sigma(z_1)\psi_\sigma(z_2)\rangle \sim (z_1 - z_2)^{-2\sigma}$
- these are the building blocks of the CFT
- if they satisfy suitable boundary conditions on $\partial\mathcal{D}$ then the whole scaling theory is conformally covariant in a strict sense under mappings $\Phi : \mathcal{D} \to \mathcal{D}'$

Integrability I: Yang-Baxter Equations

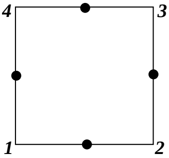




- transfer matrices $\mathbf{t}(u)$ for different u commute
- weights $W(u)$ are analytic in spectral parameter u
- assuming this lifts to the analyticity of the eigenvalues $\Lambda(u)$ of $\mathbf{t}(u)$, Baxter and others were able to deduce many consequences including the values of scaling dimensions
- these agree with the corresponding CFT

Analyticity IV: Discrete Holomorphicity

- $\mathcal{G}$ is a planar graph (*e.g.* square lattice) embedded in $\mathbf{R}^2$
- $F(z_{jk})$ is a function defined on the mid-points z_{jk} of the edges (jk) of $\mathcal{G}$
- F is *discretely holomorphic* if



$$F(z_{12}) + iF(z_{23}) + i^2F(z_{34}) + i^3F(z_{41}) = 0$$ around each square

- discrete version of Cauchy's theorem

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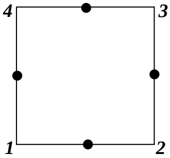

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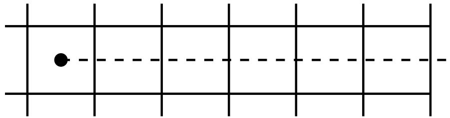
- discrete version of Cauchy's theorem
- Warning there are only N_{faces} linear equations for N_{edges} unknowns – not enough to determine $F(z_{jk})$ – additional arguments are needed to assert that F becomes an analytic function in the continuum limit

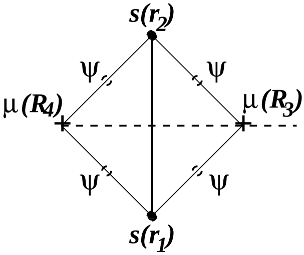
Example: the Ising model

- many $2d$ lattice models possess a duality symmetry: order $s(r) \leftrightarrow$ disorder $\mu(R)$
- for the nearest neighbor Ising model with $\mathcal{H}(\{s\}) = -\sum_{rr'} J_{rr'} s(r)s(r')$, $\mu(R)$ corresponds to $J_{rr'} \to -J_{rr'}$ on edges (rr') which cross a 'string' attached to R:



- define parafermion $\psi_\sigma(rR)$ on the edge (rR):

$$\psi_\sigma(rR) = s(r) \cdot \mu(R)\, e^{-i\sigma\theta(rR)}$$



$$\left(1 + (\tanh J_y) s(r_1) s(r_2)\right) \mu(R_4) = \left(1 - (\tanh J_y) s(r_1) s(r_2)\right) \mu(R_3)$$

- multiply both sides by $s(r_1)$ and $s(r_2)$ and use $s^2 = 1$:
- linear equations in neighboring ψ_σs $\Rightarrow$ ψ_σ is discretely holomorphic as long as:
  - $s = \frac{1}{2}$ (for the Ising model)
  - we distort the square into a rhombus (whose angle depends on J_y/J_x)
  - these lie on the *critical manifold* $\sinh J_x \sinh J_y = 1$

Discrete Holomorphicity and Integrability

- it is also possible to define ψ_σ in terms of the *loop representation* of the model (*e.g.* spin cluster boundaries)
- many (all?) lattice models with discrete states (even those without a duality symmetry) have loop representations, for which we can define *parafermionic observables* ψ_σ
- as long as these are defined suitably, these turn out, in all known cases [Smirnov, (Riva,Rajabpour,Ikhlef)+JC] to be discretely holomorphic, as long as:
  - σ is chosen suitably
  - the weights of the model are critical

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discrete holomorphicity $\Rightarrow$ integrable criticality

Universality, Integrability and Analyticity

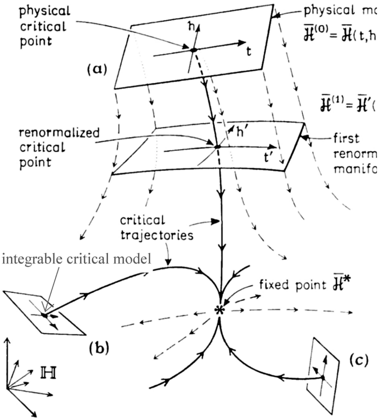

Figure: E.g. (b) is an integrable critical model, and the RG flows $(b) \rightarrow$ fixed point are special, preserving (discrete) analyticity

Can we learn anything from this (e.g.) in higher dimensions?

- role of analyticity
  - *what* should be analytic in *what*?
- role of integrability
  - what kind of integrable structures?
- role of universality
  - what models are special in each universality class?

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Joel!!