Analyticity I: Landau Theory

- **Landau free energy**

\[ F[M] = \int \left( (\nabla M)^2 + \text{Tr}(H \cdot M) + r_0 \text{Tr}M^2 + \lambda_3 \text{Tr}M^3 + \cdots \right) d^d x \]

- **all analytic terms allowed by symmetry**

- \( \delta F/\delta M = 0 \Rightarrow \textit{critical points} \) at special values of \( \{\lambda_j\} = (H, r_0, \lambda_3, \ldots) \)

- **singular behavior from bifurcations**

- \( \textit{critical exponents} \) describing how \( M \) behaves close to critical points are \textit{super-universal}.\]
Analyticity II: Renormalization Group

\[ Z[\mathbf{H}] = \int_{|q|<\Lambda} [d\mathbf{M}(x)] e^{-F[\mathbf{M}]} \]

\[ \frac{d\lambda_j}{d\ell} = -\Lambda \frac{\partial \lambda_j}{\partial \Lambda} = -\beta_j(\{\lambda\}) \]

- **fixed points** \( \beta_j(\{\lambda^*\}) = 0 \)
- \( \beta_j(\{\lambda\}) \) assumed to be analytic in neighborhood
- singular behavior from infinite iterations of an analytic mapping
- critical exponents and other universal critical properties given by derivatives of \( \beta \)-functions
- scaling fields \( \langle \phi_j(r_1)\phi_j(r_2) \rangle \sim |r_1 - r_2|^{-2x_j} \)

The only evidence we have for this picture being precisely correct for non-trivial cases is

- perturbative analysis about a trivial fixed point (e.g. \( \epsilon \)-expansion)
- integrable lattice models in \( d = 2 \)
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Universality I: Renormalization Group

Figure: M E Fisher, Rev. Mod. Phys. 70 653 (1998) [and http://terpconnect.umd.edu/~xpectnil/]
Analyticity III: Conformal Field Theory

- at (isotropic) $2d$ RG fixed points there are special scaling fields whose correlation functions are holomorphic (analytic) functions of $z = x + iy$ (or $\bar{z} = x - iy$):

- conserved currents corresponding to symmetries, e.g. stress tensor $(T(z), \bar{T}(\bar{z}))$

- parafermionic fields $\psi_\sigma(z)$ with $\langle \psi_\sigma(z_1)\psi_\sigma(z_2) \rangle \sim (z_1 - z_2)^{-2\sigma}$

- these are the building blocks of the CFT

- if they satisfy suitable boundary conditions on $\partial D$ then the whole scaling theory is conformally covariant in a strict sense under mappings $\Phi : D \rightarrow D'$
Integrability I: Yang-Baxter Equations

\[ u \rightarrow v = \frac{u}{u-v} \]

- transfer matrices \( t(u) \) for different \( u \) commute
- weights \( W(u) \) are analytic in spectral parameter \( u \)
- assuming this lifts to the analyticity of the eigenvalues \( \Lambda(u) \) of \( t(u) \), Baxter and others were able to deduce many consequences including the values of scaling dimensions
- these agree with the corresponding CFT
Analyticity IV: Discrete Holomorphicity

- $G$ is a planar graph (e.g. square lattice) embedded in $\mathbb{R}^2$
- $F(z_{jk})$ is a function defined on the mid-points $z_{jk}$ of the edges $(jk)$ of $G$
- $F$ is \textit{discretely holomorphic} if

\[
F(z_{12}) + iF(z_{23}) + i^2F(z_{34}) + i^3F(z_{41}) = 0
\]
around each square

- discrete version of Cauchy’s theorem
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- discrete version of Cauchy’s theorem
- \textbf{Warning} there are only $N_{\text{faces}}$ linear equations for $N_{\text{edges}}$ unknowns – not enough to determine $F(z_{jk})$ – additional arguments are needed to assert that $F$ becomes an analytic function in the continuum limit
Example: the Ising model

- many 2d lattice models possess a duality symmetry: order $s(r) \leftrightarrow$ disorder $\mu(R)$

- for the nearest neighbor Ising model with
  $\mathcal{H}(\{s\}) = -\sum_{rr'} J_{rr'} s(r) s(r')$, $\mu(R)$ corresponds to
  $J_{rr'} \rightarrow -J_{rr'}$ on edges $(rr')$ which cross a ‘string’ attached to $R$:

- define parafermion $\psi_\sigma(rR)$ on the edge $(rR)$:
  $$\psi_\sigma(rR) = s(r) \cdot \mu(R) e^{-i\sigma \theta(rR)}$$
\[
(1 + (\tanh J_y)s(r_1)s(r_2)) \mu(R_4) = (1 - (\tanh J_y)s(r_1)s(r_2)) \mu(R_3)
\]

- multiply both sides by \( s(r_1) \) and \( s(r_2) \) and use \( s^2 = 1 \):
- linear equations in neighboring \( \psi_\sigma s \Rightarrow \psi_\sigma \) is discretely holomorphic as long as:
  - \( s = \frac{1}{2} \) (for the Ising model)
  - we distort the square into a rhombus (whose angle depends on \( J_y/J_x \))
  - these lie on the critical manifold \( \sinh J_x \sinh J_y = 1 \)
it is also possible to define $\psi_\sigma$ in terms of the loop representation of the model (e.g. spin cluster boundaries)

many (all?) lattice models with discrete states (even those without a duality symmetry) have loop representations, for which we can define *parafermionic observables* $\psi_\sigma$

as long as these are defined suitably, these turn out, in all known cases [Smirnov, (Riva, Rajabpour, Ikhlef)+JC] to be discretely holomorphic, as long as:

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\begin{align*}
\text{discrete holomorphicity} & \Rightarrow \text{integrable criticality}
\end{align*}
Universality, Integrability and Analyticity

Figure: E.g. (b) is an integrable critical model, and the RG flows \((b) \rightarrow \text{fixed point}\) are special, preserving (discrete) analyticity.
Can we learn anything from this (e.g.) in higher dimensions?

- role of analyticity
  - what should be analytic in what?
- role of integrability
  - what kind of integrable structures?
- role of universality
  - what models are special in each universality class?
Congratulations on your 100th Statistical Mechanics Conference Joel!!