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NON-LINEAR SIGMA MODELS
A RETROSPECTIVE LOOK

to Joel, with gratitude and affection...

E. Brézin (ENS, Paris)

Where does the name come from ?

Gell-Mann and Lévy (1960)

- A fermionic isodoublet of massless fermions (!)
- An isotriplet of pseudoscalar bosons (pions)
- A scalar field σ

The interaction

$$\mathcal{L} = g\bar{\psi}(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\psi - \mu^2(\sigma^2 + \vec{\pi}^2) + \lambda(\sigma^2 + \vec{\pi}^2)^2$$

is invariant under $SU(2) \times SU(2)$

- $$\psi_R \rightarrow U\psi_R$$
- $$\psi_L \rightarrow V\psi_L$$
- $$(\sigma + i\vec{\pi}\cdot\vec{\tau}) \rightarrow V(\sigma + i\vec{\pi}\cdot\vec{\tau})U^{-1}$$

Since $SU(2) \times SU(2)$ is isomorphic to O(4), $(\sigma, \vec{\pi})$ transforms as a vector under O(4).

Under spontaneous symmetry breaking

$$< \sigma >= v \neq 0$$

the pions become massless (Goldstone bosons) and the fermions acquire the mass gv.

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INTERACTION OF GOLDSTONE PARTICLES IN TWO DIMENSIONS. APPLICATIONS TO FERROMAGNETS AND MASSIVE YANG-MILLS FIELDS

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Interaction of Goldstone particles in two dimensions lead to the infrared catastrophe. In order to analyze it we apply to the problem the method of the renormalization group. It is shown that due to interaction the regime of the "asymptotic freedom" arises. The continuation to higher dimensions and the applications of the result are briefly discussed.

"In spite of "

N.D. Mermin, H. Wagner: "Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models", Phys. Rev. Lett. 17, 11331136 (1966)

Sidney Coleman: "There are no Goldstone bosons in two dimensions", Commun. Math. Phys. 31, 259 (1973)

PHYSICAL REVIEW LETTERS 36, March '76

Renormalization of the non-linear sigma model in 2+ϵ dimension. Application to the Heisenberg ferromagnets

E.B and J. Zinn-Justin

The nonlinear σ model is renormalizable and asymptotically free in two dimensions. We show here how to construct this model in $2+\epsilon$ dimensions. Renormalization-group equations follow and exhibit a non-trivial UV stable fixed point. The existence of systematic expansions in powers of $(d-2)$ follows from this analysis. ...

The model is the continuum limit of the classical n-vector model :

$$H/kT = \frac{J}{kT} \sum_{n.n.} (\vec{S}_i - \vec{S}_j)^2$$

with

$$\vec{S}^2 = 1$$

The parametrization of the sphere

$$\vec{S} = \{\sqrt{(1 - T\vec{\pi}^2)}, \sqrt{T}\vec{\pi}\}$$

leads to an expansion in powers of T.

Several differences with the "usual" φ^4 :

- A naive UV cut-off (on the $\vec{\pi}$ field) breaks the $O(n)$ - symmetry, and there is an infinite number of marginal interactions :
$$(\pi)^k \nabla\pi \cdot \nabla\pi$$
- Infrared singularities "regularized" by an external magnetic field
$$\frac{1}{T} h\sigma(x) = -\frac{1}{2} h\vec{\pi}^2 + O(\pi^4)$$
which breaks the O(n) symmetry adds new interactions $h\pi^{2k}$.

- **The coupling constant T is a relevant variable** : hence the necessity of an IR-unstable (i.e. UV stable) fixed point.

$$\beta(T) = \epsilon T - (n-2)T^2[1 + T + \frac{1}{4}(n+2)T^2$$

$$+\{\frac{3}{2}\zeta(3)(n-3) - \frac{1}{12}(n^2 - 22n + 34)\}T^3 + 0(T^4)]$$

$$(d-2)\nu = 1 - \frac{\epsilon}{n-2} + \frac{4-n}{2(n-2)^2}\epsilon^2$$

$$+\frac{n^2 - 10n + 18 + 18(3-n)\zeta(3)}{4(n-2)^3}\epsilon^3 + O(\epsilon^4)$$

... not very efficient for 3D and $n = 3$, but the coherence with $1/n$ expansion near two and four dimensions indicates that the whole (n,d) space was under control.

PHYSICAL REVIEW LETTERS 45, September '80

Nonlinear Models in $2+\epsilon$ dimensions

D.Friedan

A generalization of the nonlinear σ model is considered. The field takes values in a compact manifold M and the coupling is determined by a Riemannian metric on M. The model is renormalizable in $2+\epsilon$ dimensions, the renormalization group acting on the infinite dimensional space of Riemannian metrics. ...

For $d=2$

$$S\{\varphi\} = \frac{1}{T}\int d^2x g_{ij}(x)\partial_\mu\varphi^i\partial_\mu\varphi^j$$

$$\beta_{ij}\{T^{-1}g\} = R_{ij} + O(R^2)$$

with the Ricci tensor

$$R_{ij} = R^k_{ikj}$$

Nonlinear sigma models were introduced as string propagation in backgroung fields. Conformal invariance required a manifold with zero Ricci curvature (Calabi-Yau manifolds).

Zeitschrift fur Physik B 35, '79

The mobility edge problem : Continuous symmetry and a conjecture

Franz Wegner

An apparently overlooked symmetry of the disordered electron problem is derived. It yields the well-known Ward-identity connecting the one- and two-particle Green's function. This symmetry and the apparent shortrange behaviour of the averaged one-particle Green's function are used to conjecture that the critical behaviour near the mobility edge coincides with that of interacting matrices which have two different eigenvalues of multiplicity zero (due to replicas)...

The averaging over disorder of

$$< r_1|\frac{1}{z_1 - H}|r_1' >< r_2|\frac{1}{z_2 - H}|r_2' >$$

leads to a sigma model for the coset

$$\frac{U(n_1 + n_2)}{U(n_1) \times U(n_2)}$$

with

$$n_1, \quad , n_2 \to 0$$

The model involves a matrix Q of size $(n_1 + n_2) \times (n_1 + n_2)$ and an action

$$A\{Q\} = \int d^d x \mathrm{Tr} \nabla Q \cdot \nabla Q$$

which satisfies the non-linear constraint

$$Q^2 = 1$$

and can be written as

$$Q = T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T$$

in which $T \in U(n_1 + n_2)$

Nuclear Physics B(1980) 528-544

GENERALIZED NON-LINEAR σ-MODELS WITH GAUGE INVARIANCE

E.B., S. Hikami and J. Zinn-Justin

Generalizations of the non-linear σ-model and of the CP^N model are considered. They are invariant under a global group G and under a local gauge subgroup H. Whenever G/H defines a symmetric space, we can show that these models are asymptotically free. .... Then quantization is considered for the models $U(N)/U(p) \times U(N-p)$ and for their orthogonal analogue. Two-loop results for the renormalization group functions are given.

The first application of this approach was a derivation of the renormalization-group (RG) equations of the scaling theory of Anderson localization and led to an expansion near two-dimensions of the Anderson et al. transition for non-interacting electrons.

Starting from the seminal papers of Wegner and Efetov , a field-theoretical description based on the nonlinear σ model has become one of the main analytical approaches to various problems in disordered electronic systems. The ensemble averaging over all configurations of disorder is performed either using bosonic or fermionic n-replicated fields and taking the $n \rightarrow 0$ limit in the results, or using supersymmetric fields.

Semiclassical Field Theory Approach to Quantum Chaos

A. V. Andreev, B. D. Simons, O. Agam, and B. L. Altshuler

We construct a field theory to describe energy averaged quantum statistical properties of systems which are chaotic in their classical limit. An expression for the generating function of general statistical correlators is presented in the form of a functional supermatrix nonlinear σ model where the effective action involves the evolution operator of the classical dynamics....

Topological terms, instantons, θ-vacuum

A. D´ Adda, P. Di Vecchia and M. Luescher,

Nucl. Phys. B 146, 63 (1978)

CP^{n-1} models : generalization of sigma-models .

One introduces n-complex fields $z_a(x)$ with the non-linear constraint

$$|z_1|^2 + \cdots |z_n|^2 = 1$$

and the action

$$A = \int d^2x[\partial_\mu \bar{z}_a \partial_\mu z_a + (\bar{z}_a \partial_\mu z_a)^2]$$

The ordinary O(3) sigma-model is simply the case n=2 , with

$$S^a = \bar{z}_\alpha \tau^a_{\alpha\beta} z_\beta$$

This CP^{n-1} models possess instanton solutions for any n.

One can introduce a topological density

$$q(x) = \frac{i}{2\pi} \epsilon_{\mu\nu} \partial_\mu \bar{z} \partial_\nu z$$

and

$$Q = \int d^2 x q(x)$$

is an integer.

One can thus change the initial Boltzmann weight $\exp -A \rightarrow \exp -A + i\theta Q$ and one expects periodicity in θ with period 2π.

This possibility of a θ term in sigma models was advocated in 1983 by Levine, Libby, Pruisken in the theory of the (integer) quantum Hall effect

$$S[Q] = -g \int d^2x \mathrm{Tr} \partial_\mu \mathrm{Q} \partial_\mu \mathrm{Q} + \sigma_{\mathrm{H}} \int \mathrm{d^2x Tr} \epsilon_{\mu\nu} \mathrm{Q} \partial_\mu \mathrm{Q} \partial_\nu \mathrm{Q}$$

with a topological charge

$$C[Q] = \frac{1}{16\pi} \int d^2x \mathrm{Tr} \epsilon_{\mu\nu} \mathrm{Q} \partial_\mu \mathrm{Q} \partial_\nu \mathrm{Q}$$

coupled to the Hall conductance σ_{xy}.

The role of this topological charge for the theory of the (integer) quantum Hall effect has remained controversial. In a recent article Pruisken has reanalyzed the contradiction between the large N picture of the CP^N model and the instanton picture and argued that the topological term is responsible, for any N, of edge currents which have not been considered in the bulk treatment.

Field theory remains in many aspects a component of statistical physics.

Beyond σ models, the remarkable advances due to 2D-conformal field theories, there are now even claims that Maldacena duality can be used for understanding quantum critical points (Herzog et al, Phys Rev 2007)?