Exact solution of the six-vertex model with domain wall boundary conditions

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100th SM Conference of Joel Lebowitz

Rutgers University

December 14, 2008

Introduction

The six-vertex model, or the model of two-dimensional ice, is stated on a square lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such rule is sometimes called the ice-rule. There are only six possible configurations of arrows at each vertex, hence the name of the model.

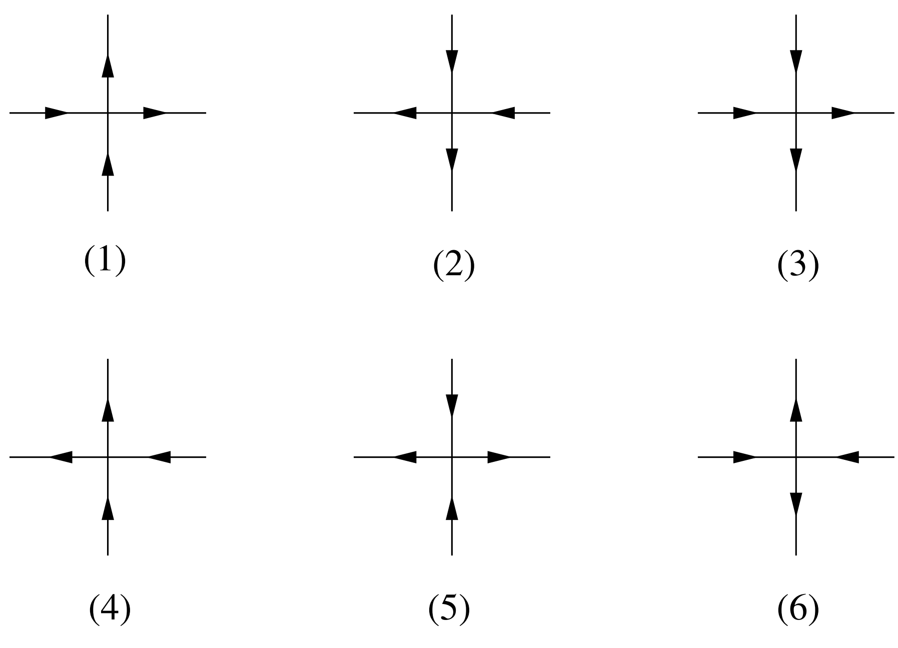


Fig. 1. The six possible configurations of arrows at each vertex.

We will consider the domain wall boundary conditions (DWBC) in which the arrows on the upper and lower boundaries point in the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice is shown on Fig. 2.

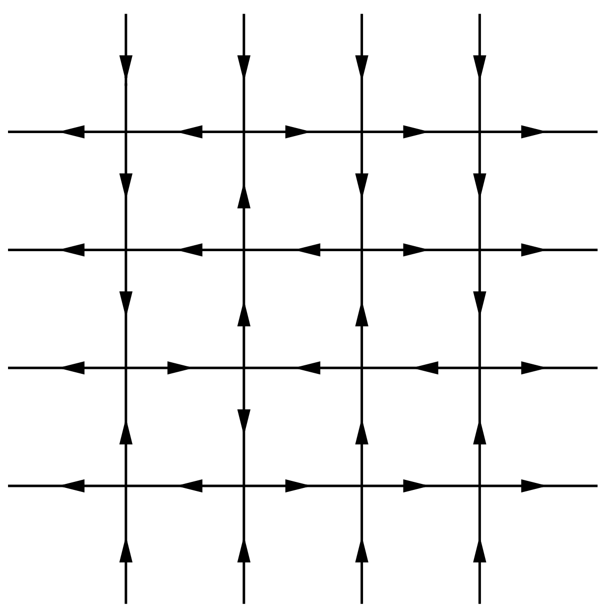


Fig. 2. An example of 4×4 configuration.

The name of square ice comes from the two-dimensional arrangement of water molecules, H_2O, with oxygen atoms at the vertices of a lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them.

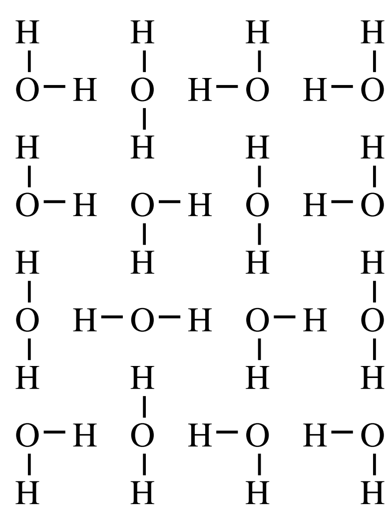


Fig. 3. The corresponding ice model.

For each possible vertex state we assign the weight w_i, $i = 1, \ldots, 6$, and define, as usual, the partition function as a sum of the configuration weights over all possible arrow configurations, with a configuration weight given as a product of the corresponding vertex weights:

$$Z_N = \sum_{\text{arrow configurations}} w(\sigma),$$

$$w(\sigma) = \prod_x w_{\sigma(x)} = \prod_{i=1}^{6} w_i^{n_i(\sigma)},$$

where $\sigma(x)$ is the vertex configuration number of σ at the vertex x and $n_i(\sigma)$ is the number of vertices in the state i in σ. Here σ is a configuration on an $N \times N$ square lattice.

The Gibbs measure:

$$\mu_N(\sigma) = \frac{w(\sigma)}{Z_N}.$$

Problems

We will be interested in two problems:

1. What is the large N asymptotic behavior of Z_N?

2. What are typical configurations σ with respect to the Gibbs measure, as $N \to \infty$?

The six-vertex model has six parameters: the weights w_i. By using some conservation laws it can be reduced to only two parameters.

Conservation laws

Proposition 1. *For any configuration σ of the six vertex model with DWBC, we have that*

$$n_1(\sigma) = n_2(\sigma),$$

and

$$n_3(\sigma) = n_4(\sigma).$$

Also,

$$n_5(\sigma) - n_6(\sigma) = N.$$

The conservation laws allow to reduce the weights $w_1, \ldots, w_6$ to 3 parameters. Namely, we have that

$$w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} w_5^{n_5} w_6^{n_6} = C_N a^{n_1} a^{n_2} b^{n_3} b^{n_4} c^{n_5} c^{n_6},$$

where

$$a = \sqrt{w_1 w_2}, \quad b = \sqrt{w_3 w_4}, \quad c = \sqrt{w_5 w_6},$$

and the constant

$$C_N = \left(\frac{w_5}{w_6} \right)^{\frac{N}{2}} .$$

This implies the relation between the partition functions,

$$Z_N(w_1, w_2, w_3, w_4, w_5, w_6) = C_N Z_N(a, a, b, b, c, c),$$

and between the Gibbs measures,

$$\mu_N(\sigma; w_1, w_2, w_3, w_4, w_5, w_6) = \mu_N(\sigma; a, a, b, b, c, c).$$

Therefore, for the DWBC, the general weights are reduced to the case when

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c.$$

Furthermore,

$$Z_N(a, a, b, b, c, c) = c^{N^2} Z_N\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right)$$

and

$$\mu_N(\sigma; a, a, b, b, c, c) = \mu_N\left(\sigma; \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right),$$

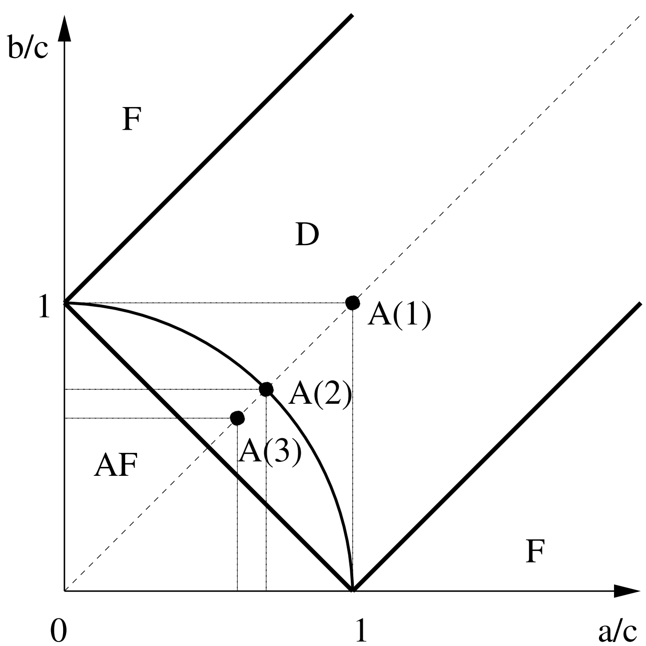
so that the general weights reduce to the two parameters, $\frac{a}{c}, \frac{b}{c}$.

Phase Diagram

Introduce the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

The phase diagram of the six-vertex model consists of the three phase regions: ferroelectric phase region, $\Delta > 1$; the anti-ferroelectric phase region, $\Delta < -1$; and, the disordered phase region, $-1 < \Delta < 1$.



In these phase regions we parameterize the weights in the standard way: in the ferroelectric phase region,

$$a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma), \quad c = \sinh(2\gamma), \\ |\gamma| < t,$$

in the anti-ferroelectric phase region,

$$a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \\ |t| < \gamma,$$

and in the disordered phase region

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \\ |t| < \gamma.$$

Large N-asymptotics of the partition function

Disordered phase region

Theorem 2. (Bleher and Fokin, 2006). *There exists $\varepsilon > 0$ such that*

$$Z_N = CN^\kappa F^{N^2} \left(1 + O(N^{-\varepsilon})\right),$$

where $C > 0$,

$$F = \frac{\pi[\cos(2t) - \cos(2\gamma)]}{4\gamma \cos \frac{\pi t}{2\gamma}},$$

and

$$\kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}.$$

Pavel Bleher and Vladimir Fokin, Exact solution of the six-vertex model with domain wall boundary condition. Disordered phase. *Commun. Math. Phys.* **268** (2006), 223–284.

The six-vertex model with periodic boundary conditions (PBC) was found by Lieb by means of the Bethe Ansatz,

E. H. Lieb, *Phys. Rev. Lett.* **18** (1967) 692; *Phys. Rev. Lett.* **18** (1967) 1046-1048; *Phys. Rev. Lett.* **19** (1967) 108-110; *Phys. Rev.* **162** (1967) 162.

See also

B. Sutherland, *Phys. Rev. Lett.* **19** (1967) 103-104,

and

E. H. Lieb and F. Y. Wu, Two dimensional ferroelectric models, in *Phase Transitions and Critical Phenomena*, C. Domb and M. Green eds., vol. 1, Academic Press (1972) 331-490.

It is interesting to notice that the free energy with DWBC and the one with PBC are different.

Ferroelectric Phase Region

(joint work with Karl Liechty)

Ferroelectric phase region: $b > a + c$ or $a > b + c$. We will assume $b > a + c$.

Parametrization:

$$a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma),$$
$$c = \sinh(2\gamma), \quad 0 < \gamma < t.$$

Theorem 3. *In the ferroelectric phase region with* $t > \gamma > 0$*, for any* $\varepsilon > 0$*, as* $N \to \infty$*,*

$$Z_N = CG^N F^{N^2} \left[1 + O\left(e^{-N^{1-\varepsilon}}\right)\right],$$

where $C = 1 - e^{-4\gamma}$*,* $G = e^{\gamma - t}$*, and* $F = b = \sinh(t + \gamma)$*.*

Reference

P.M. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary conditions. Ferroelectric phase. arXiv:0712.4091 [math.ph]; (to appear in CMP)

Critical Line between Disordered and Ferroelectric Phase Regions

Consider now the critical line: $b = a + c$.

Parameterization:

$$b = \frac{\alpha + 1}{2}, \quad a = \frac{\alpha - 1}{2}, \quad c = 1.$$

Theorem 4. *On the critical line, as* $N \to \infty$,

$$Z_N = C N^{\kappa} G^{\sqrt{N}} F^{N^2} [1 + O(N^{-1/2})] \,,$$

where $C > 0$, $\kappa = \frac{1}{4}$, *and*

$$G = \exp\left[-\zeta\left(\frac{3}{2}\right) \sqrt{\frac{a}{\pi}} \right], \quad F = b\,.$$

Reference

P.M. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary conditions. Critical line between disordered and ferroelectric phases.
arXiv:0802.0690 [math.ph]; (to appear in JSP).

Sketch of the Proofs

Disordered Phase

The formula of Izergin and Korepin:

$$Z_N = \frac{(ab)^{N^2}}{\left(\prod_{n=0}^{N-1} n!\right)^2} \tau_N \,,$$

where τ_N is the Hankel determinant,

$$\tau_N = \det \left(\frac{d^{i+k-2}\phi}{dt^{i+k-2}} \right)_{1 \le i,k \le N} ,$$

and

$$\phi(t) = \frac{c}{ab} \,.$$

An elegant derivation of the Izergin-Korepin formula from the Yang-Baxter equations is given in the papers of Korepin and Zinn-Justin,

V. Korepin and P. Zinn-Justin, *J. Phys. A* **33** No. 40 (2000), 7053

and Kuperberg,

G. Kuperberg, *Intern. Math. Res. Notes* (1996), 139-150.

The Zinn-Justin formula

Suppose that

$$\phi(t) = \int_{-\infty}^{\infty} e^{tx} m(x)\, dx.$$

Then

$$\tau_N = \frac{1}{N!} \int \Delta(\lambda)^2 \prod_{j=1}^{N} w(\lambda_j) d\lambda,$$

where $\Delta(\lambda)$ is the Vandermonde determinant,

$$\Delta(\lambda) = \det(\lambda_j^{k-1})_{1 \le j,k \le N}$$

and

$$w(x) = e^{tx} m(x).$$

Reduction to orthogonal polynomials

Consider monic polynomials $P_n(x) = x^n + \dots$, orthogonal on the line with respect to the weight $w(x) = e^{tx}m(x)$, so that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)e^{tx}m(x)dx = h_n\delta_{nm}.$$

Then the Zinn-Justin formula implies that

$$\tau_N = \prod_{n=0}^{N-1} h_n.$$

The orthogonal polynomials satisfy the three term recurrent relation,

$$xP_n(x) = P_{n+1}(x) + Q_n P_n(x) + R_n P_{n-1}(x),$$

where R_n can be found as $R_n = \frac{h_n}{h_{n-1}}$, This gives that $h_n = h_0 \prod_{j=1}^{n} R_j$, where

$$h_0 = \int_{-\infty}^{\infty} e^{tx} m(x) dx = \frac{\sin(2\gamma)}{\sin(\gamma + t)\sin(\gamma - t)}.$$

Thus,

$$\tau_N = h_0^N \prod_{n=1}^{N-1} R_n^{N-n}.$$

Weight for the disordered phase region

Let

$$m(x) = \frac{\sinh(\frac{\pi}{2} - \gamma)x}{\sinh \frac{\pi x}{2}}.$$

Then

$$\phi(t) = \int_{-\infty}^{\infty} e^{tx} m(x)\, dx,$$

hence the weight for the orthogonal polynomials in the disordered phase region is

$$w(x) = \frac{e^{tx} \sinh(\frac{\pi}{2} - \gamma)x}{\sinh \frac{\pi x}{2}}.$$

The main technical result of our work with Fokin is the asymptotics of R_n as $n \to \infty$.

Theorem 5. *(Asymptotics of the recurrent coefficient). As $n \to \infty$,*

$$R_n = \frac{n^2}{\gamma^2}[R + \cos(n\omega) \sum_{j:\, \kappa_j \le 2} c_j n^{-\kappa_j} + cn^{-2} + O(n^{-2-\varepsilon})], \qquad \varepsilon > 0,$$

where

$$R = \left(\frac{\pi}{2\cos\frac{\pi\zeta}{2}}\right)^2, \quad \zeta \equiv \frac{t}{\gamma}; \qquad \omega = \pi(1+\zeta);$$

$$\kappa_j = 1 + \frac{2j}{\frac{\pi}{2\gamma} - 1},$$

and c_j, c are some explicit numbers.

The proof of the theorem is based on the matrix Riemann-Hilbert problem for orthogonal polynomials and nonlinear steepest descent method.

Ferroelectric phase: Discrete orthogonal polynomials

Introduce discrete monic polynomials $P_j(x) = x^j + \dots$ orthogonal on the set $\mathbb{N} = \{1, 2, \dots\}$ with respect to the weight,

$$w(l) = 2e^{-2tl} \sinh(2\gamma l)$$
$$= e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l} = q^l - r^l,$$

$$1 > q = e^{-2t+2\gamma} > r = e^{-2t-2\gamma} > 0.$$

so that

$$\sum_{l=1}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}.$$

Then

$$\tau_N = 2^{N^2} \prod_{k=0}^{N-1} h_k,$$

Meixner Polynomials

For normalized Meixner polynomials we have

$$\sum_{l=1}^{\infty} Q_j(l)Q_k(l)q^l = h_k^{\mathrm{Q}}\delta_{jk},$$

$$h_k^{\mathrm{Q}} = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}}.$$

We prove the following asymptotics of h_k.

Theorem 6. *For any $\varepsilon > 0$, as $k \to \infty$,*

$$h_k = h_k^{\mathrm{Q}} \left(1 + O(e^{-k^{1-\varepsilon}})\right).$$

This implies the required asymptotics of Z_N up to a constant factor. To get the constant we use the Toda equation for deformations of the partition function and the asymptotics of Z_N as $t \to \infty$.

Critical Line between Ferroelectric and Disordered Phase Regions

Partition function:

$$Z_N = \left(\frac{b}{c}\right)^{N^2} \prod_{k=0}^{N-1} \frac{h_k}{(k!)^2}\,,.$$

where

$$\int_0^\infty P_j(x)P_k(x)w(x)dx = h_k\delta_{jk}\,;$$

$$w(x) = e^{-x} - e^{-rx},$$

$$r = \frac{b}{a} > 1.$$

Our main technical result is the following asymptotics of h_k.

Theorem 7. *As $k \to \infty$,*

$$\ln\left[\frac{h_k}{(k!)^2}\right] = -\frac{\zeta(\frac{3}{2})}{2\sqrt{\pi(r-1)}k^{1/2}} + \frac{1}{4k} + O(k^{-3/2}).$$

To prove the theorem we use the Vanlessen's asymptotic formula. The main difficulty is that $w(z) = e^{-z} - e^{-rz}$ has zeros on the imaginary axis. Under the scaling $u = \frac{z}{N}$ the zeros accumulates to the origin as $N \to \infty$, and the Vanlessen's formula is not directly applicable. We overcome this difficulty by introducing an additional transformation of the undressed Riemann-Hilbert problem (the second undressing transformation). This enables us to justify the Vanlessen's asymptotic formula for h_k and to prove the theorem.

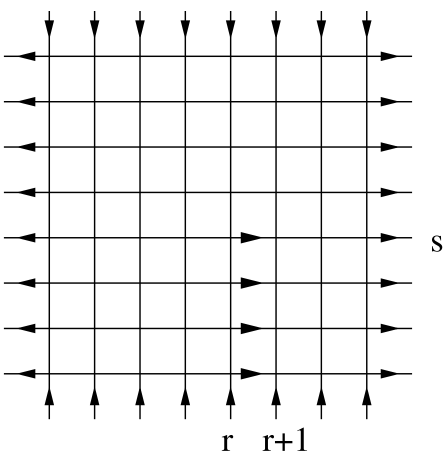
Arctic circle theorem

Joint project with Ken McLaughlin

The arctic circle type theorem describes the scaling limit of the phase separation under fixed boundary conditions. It is proven for various dimer models. Our goal is to prove the arctic circle type theorem for the six-vertex model with DWBC. Our approach is based on the Riemann-Hilbert problem and the work of Colomo and Pronko,

F. Colomo and A.G. Pronko, The arctic circle revisited. Preprint. arXiv:0704.0362.

We use the following coordinates on the lattice: $r = 1, \ldots, N$ labels the vertical lines from right to left; $s = 1, \ldots, N$ labels the horizontal lines from top to bottom.



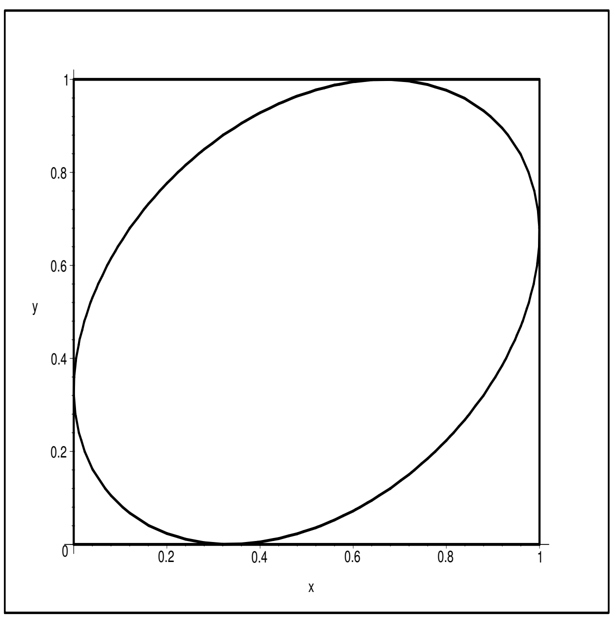
The configurations contributing to the “emptiness formation probability” are those which are consistent with the arrows shown.

Main result

We consider the free fermion line, $a^2+b^2=c^2$. Define the ellipse,

$$\mathcal{E}=\Big\{(x,y):\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 + 2\,\frac{\tau-1}{\tau+1}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)=\frac{\tau}{(\tau+1)^2}\Big\},$$

where $\tau=\frac{b^2}{a^2}$.



The ellipse $\mathcal{E}$.

Theorem 8. *(Arctic circle type theorem.) If* (x, y) *is in the right low region outside* $\mathcal{E}$ *then*

$$F_N(r, s) \geq 1 - C_1 e^{-c_1 N}, \quad (r, s) = ([Nx], [Ny])$$

and if (x, y) *is inside* $\mathcal{E}$ *then*

$$F_N(r, s) \leq C_2 e^{-c_2 N^2}.$$

This implies that the typical configurations are frozen outside of $\mathcal{E}$ and unfrozen inside $\mathcal{E}$.

Conclusion

We proved some results about the large n asymptotic behavior of the partition function of the six-vertex model with DWBC and the emptiness formation probability:

- We proved the Zinn-Justin conjecture on the large N asymptotic behavior of the partition function in the disordered phase region,

$$Z_N = CN^{\kappa} e^{N^2 F} \left(1 + O(N^{-\varepsilon})\right),$$

  and we found the explicit value of the exponent κ,

$$\kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}.$$

- In the ferroelectric phase region we proved the asymptotic formula,

$$Z_N = CG^N F^{N^2}\left[1 + O\left(e^{-N^{1-\varepsilon}}\right)\right],$$

where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma - t}$, and $F = \sinh(t+\gamma)$.

- On the critical line between disordered and ferroelectric phase regions we proved the asymptotic formula,

$$Z_N = CN^\kappa G^{\sqrt{N}} F^{N^2}[1 + O(N^{-1/2})],$$

where $C > 0$, $\kappa = \frac{1}{4}$, and

$$G = \exp\left[-\zeta\left(\frac{3}{2}\right)\sqrt{\frac{a}{\pi}}\right], \quad F = b\,.$$

- On the free fermion line we evaluated the emptiness formation probability and we obtained a proof of the arctic circle phenomenon.