Scaling function of the 2D Ising model in a magnetic field

Murray Batchelor

Department of Theoretical Physics
and Mathematical Sciences Institute
Australian National University

Rutgers, December 13, 2008

[V. Mangazeev, M. Batchelor, V. Bazhanov and M. Dudalev]

arXiv:0811.3271, J. Phys. A: Math. and Theor. FTC, in press

Outline

- 2D lattice Ising model, scaling theory
- 2D Ising field theory, scaling function
- Variational corner-transfer matrix method
- Short review of exact and known results
- Scaling function from lattice calculations

Ising model on the square lattice

$$Z = \sum_{\sigma} \exp\left\{ \beta \sum_{\langle ij \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i \right\}, \qquad \sigma_i = \pm 1$$

$$F = -\lim_{N\to\infty} \frac{1}{N} \log Z, \qquad M = -\frac{\partial F}{\partial H}, \qquad \chi = -\frac{\partial^2 F}{\partial H^2}$$

2nd order transition at $H = 0,\ \beta = \frac{1}{2}\log(1+\sqrt{2}) = 0.44068679\ldots$

$H = 0$ is exactly solvable (L. Onsager, 1944)

Scaling theory predictions (A. Aharony, M. Fisher (1980), ...)

$$F_{sing}(\tau, H) = \mathcal{F}(m, h), \quad \tau = \frac{1}{2}\left[\frac{1}{\sinh(2\beta)} - \sinh(2\beta)\right], \quad \tau \to 0, \quad H \to 0$$

$$m = m(\tau, H) = -\sqrt{2}\,\tau + O(\tau^3, H^2), \quad h = h(\tau, H) = C_h\, H + H\, O(\tau, H^2)$$

$$\boxed{\mathcal{F}(m, h) = \frac{m^2}{8\pi} \log m^2 + h^{16/15}\,\Phi(\eta), \qquad \eta = \frac{m}{h^{8/15}}}$$

2D Ising field theory

The action

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{(c=1/2)} + \frac{m}{2\pi}\int \epsilon(x)\, d^2x + h\int \sigma(x)\, d^2x$$

$h = 0$ corresponds to free-fermions and $m = 0$ leads to Zamolodchikov's integrable E_8 theory

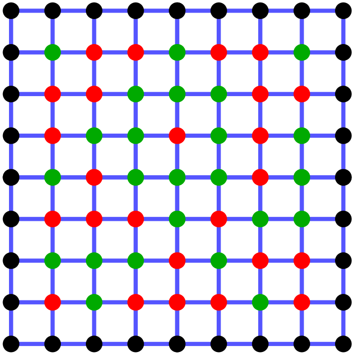
The vacuum energy density

$$\mathcal{F}(m,h) = \frac{m^2}{8\pi}\log m^2 + h^{16/15}\,\Phi(\eta), \qquad \eta = \frac{m}{h^{8/15}}$$

$\Phi(\eta)$ and its analytical properties were studied by Fonseca & Zamolodchikov (2001) using "Truncated Free-Fermion Space Approach" (TFFSA) and high- and low-T dispersion relations.

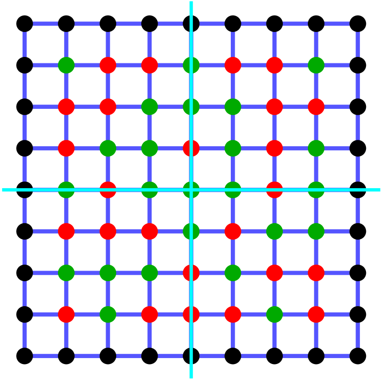
Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



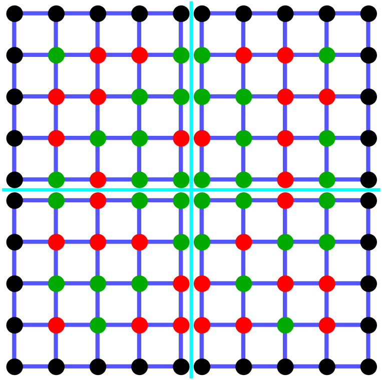
Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



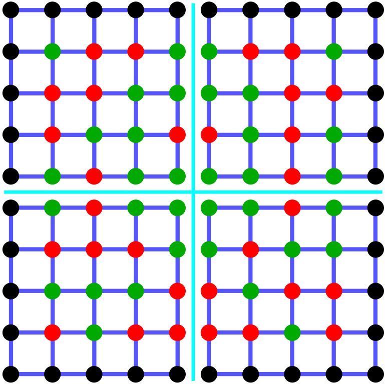
Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



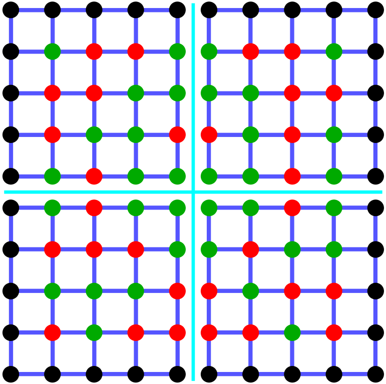
Numerical algorithm based on the corner-transfer matrix method

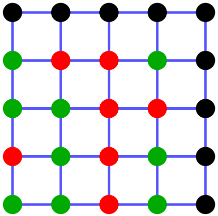
- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



Numerical algorithm based on the corner-transfer matrix method

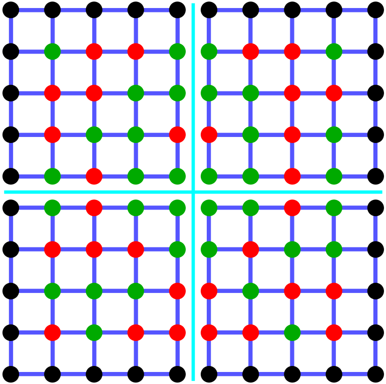
- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

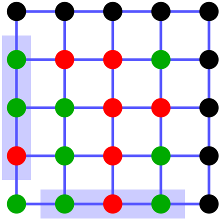




Numerical algorithm based on the corner-transfer matrix method

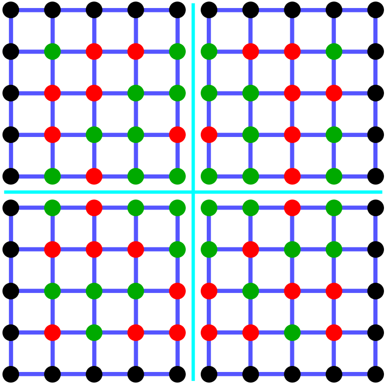
- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

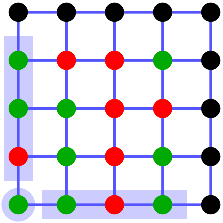




Numerical algorithm based on the corner-transfer matrix method

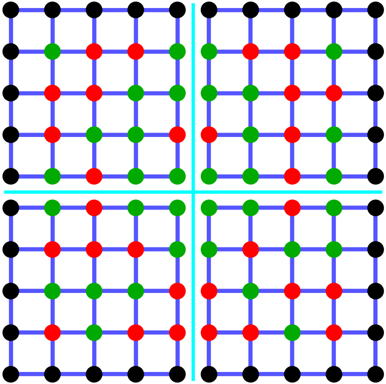
- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



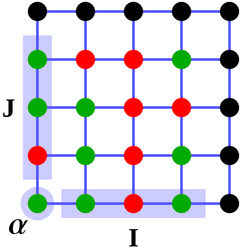


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

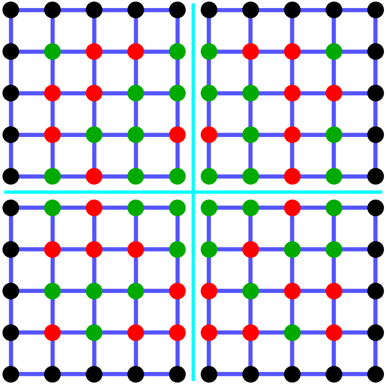


$\mathbf{I},\mathbf{J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$

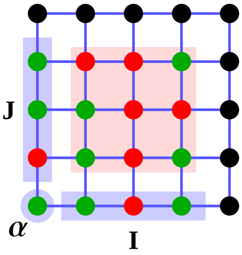


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

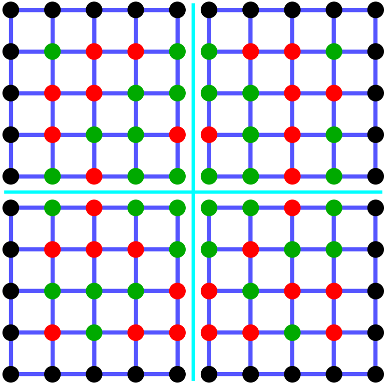


I,J$=\{\pm,\ldots,\pm\} = 1,\ldots,2^{N-1}$

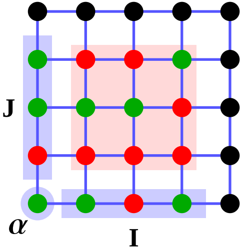


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

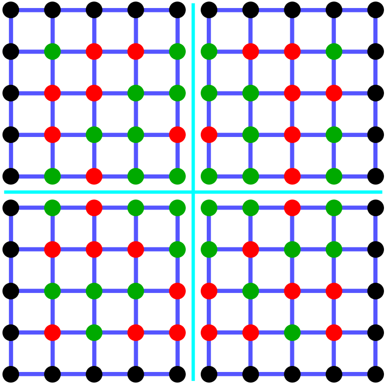


$\mathbf{I,J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$

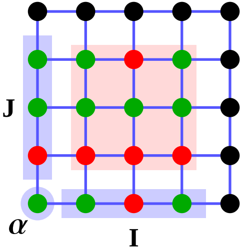


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

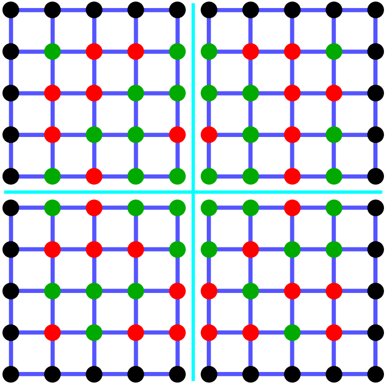


$\mathbf{I,J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$

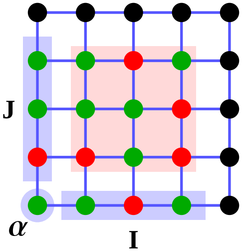


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

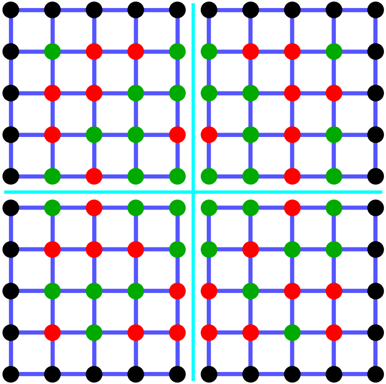


I,J$=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$

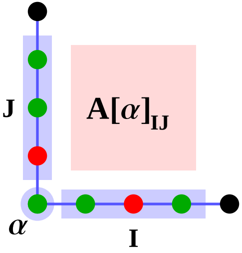


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

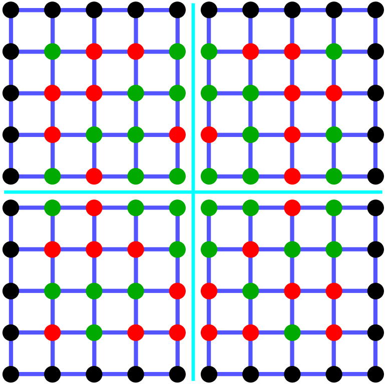


I,J$=\{\pm,\ldots,\pm\} = 1,\ldots,2^{N-1}$

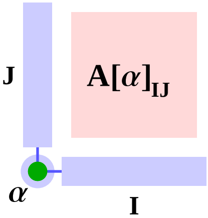


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**



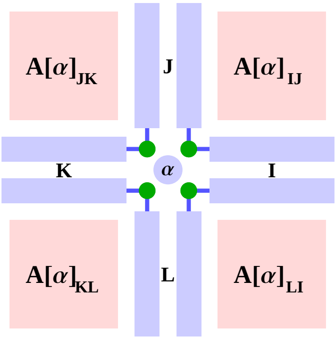
$\mathbf{I},\mathbf{J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$

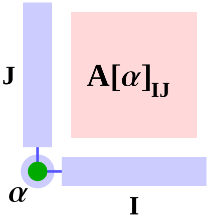


Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

$$\mathbf{I,J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$$



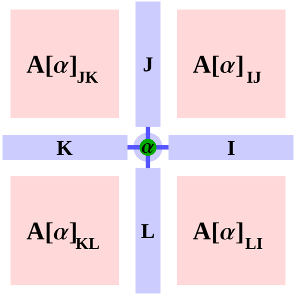


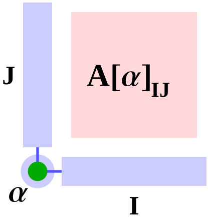
$$\mathbf{Z}=\sum_{\alpha} A[\alpha]_{IJ}A[\alpha]_{JK}A[\alpha]_{KL}A[\alpha]_{LI}=$$

Numerical algorithm based on the corner-transfer matrix method

- **The corner transfer-matrix variational method (R. Baxter, 1968, 1976)**

$$\mathbf{I,J}=\{\pm,\ldots,\pm\}=1,\ldots,2^{N-1}$$





$$\mathbf{Z}=\sum_{\alpha} A[\alpha]_{IJ}A[\alpha]_{JK}A[\alpha]_{KL}A[\alpha]_{LI}=$$

$$=\sum_{\alpha} Tr(A[\alpha]^4)$$

- The dimension of A is huge: for $N = 20$, $\dim(A) = 524388$.
- The spectrum of A exponentially decays away from the critical temperature T_c.
- Even when $T = 0.99\,T_c$, the largest 100-200 eigenvalues of A are enough to calculate physical quantities with 10^{-20} accuracy.
- We take a small size system, diagonalize A and keep only M (≈ 100) eigenvalues. We increase $N \to N + 1$, construct a new A and keep M eigenvalues.
- The size of the system becomes the number of iterations and can be as large as we wish.
- No extrapolation is needed. When the matrix A stabilizes (normally 200-300 iterations for $15 - 20$ correct digital places), we get the physical quantities at $N \to \infty$.

Renormalization group scaling

$$F(\tau, H) = F_{reg}(\tau, H) + F_{sing}(u_1, u_2, \ldots)$$

$u_j(\tau, H)$ - nonlinear scaling fields analytic in τ, H.

$$F_{sing}(u_1, u_2, \ldots) = b^{-d} F_{sing}(b^{y_1} u_1, b^{y_2} u_2, \ldots)$$

$y_i > 0$ - relevant fields, $y_i < 0$ - irrelevant fields

Ising model

$u_1 = m,\ u_2 = h,\ y_1 = 1,\ y_2 = 15/8,\ d = 2,\ y_i < 0,\ i > 2$

$$F_{sing}(u_1, u_2, \ldots) = m^2 F_{sing}(\pm, \frac{h}{|m|^{15/8}}, u_3 |m|^{|y_3|}, \ldots) \approx m^2 F^{\pm}_{sing}(\frac{h}{|m|^{15/8}})$$

$$F_{sing}(u_1, u_2, ..) = h^{16/15} F_{sing}(\frac{m}{h^{8/15}}, 1, u_3 h^{\frac{8|y_3|}{15}}, ..) \approx h^{16/15} \Phi(\frac{m}{h^{8/15}})$$

+ log corrections

More detailed structure

We know the leading log term from the Ising solution

$$\mathcal{F}(m,h) = \frac{m^2}{8\pi}\log m^2 + \begin{cases} m^2\, G_{high}(\xi), & m<0 \\ m^2\, G_{low}(\xi), & m>0 \end{cases}, \quad \xi = h/|m|^{15/8}$$

$$\mathcal{F}(m,0) = \frac{m^2}{8\pi}\log m^2, \quad G_{high}(0) = G_{low}(0) = 0$$

They can be expanded in ξ (Fonseca & Zamolodchikov)

$$G_{high}(\xi) = G_2\xi^2 + G_4\xi^4 + G_6\xi^6 + \dots$$
$$G_{low}(\xi) = \tilde{G}_1\xi + \tilde{G}_2\xi^2 + \tilde{G}_3\xi^3 + \dots$$

The expansion of $\Phi(\eta)$

$$\Phi(\eta) = -\frac{\eta^2}{8\pi}\log\eta^2 + \sum_{k=0}^{\infty}\Phi_k\eta^k$$

More detailed structure

$$\tilde{G}_1 = -2^{1/12}\, e^{-1/8}\, \mathcal{A}^{3/2} = -1.357838341706595\ldots$$

The coefficients G_2 and $\tilde{G}_2$ have integral expressions (BMW, TM) involving solutions of the Painlevé III (V) equation. They were numerically evaluated to very high precision (50 digits) in (ONGP)

$$G_2 = -1.845228078232838\ldots, \qquad \tilde{G}_2 = -0.0489532897203\ldots$$

The coefficient Φ_0 was calculated by Fateev

$$\Phi_0 = -\frac{(2\pi)^{\frac{1}{15}}\gamma(\frac{1}{3})\gamma(\frac{1}{5})\gamma(\frac{7}{15})}{\left[\gamma(\frac{1}{4})\gamma^2(\frac{3}{16})\right]^{\frac{8}{15}}} = -1.19773338379799\ldots, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

The coefficient Φ_1 has an explicit integral representation, obtained in (FLZ^2). We have evaluated the required integral explicitly

$$\Phi_1 = -\frac{32 \cdot 2^{\frac{3}{4}}}{225\,(2\pi)^{\frac{7}{15}}} \frac{\gamma(\frac{1}{3})\gamma(\frac{1}{8}) \prod_{k=3}^{7} \gamma(\frac{k}{15})}{\left[\gamma(\frac{1}{4})\gamma^2(\frac{3}{16})\right]^{\frac{19}{15}}} = -0.3188101248906\ldots$$

Lattice calculations

Using the CTM method we calculated $\approx$ 10000 high precision data points for the free energy, magnetization and internal energy in the range $10^{-7} < H < 10^{-2}$ and $0.9\beta_c < \beta < 1.1\beta_c$. Lattice free energy

$$F(\tau, H) = F_{sing}(\tau, H) + F_{reg}(\tau, H) + F_{sub}(\tau, H), \qquad \tau, H \to 0$$

$$\boxed{F_{sing}(\tau, H) = \tfrac{m^2}{8\pi} \log m^2 + h^{16/15}\,\Phi(\tfrac{m}{h^{8/15}}), \quad F_{reg}(\tau, H) = A(\tau) + H^2\,B(\tau) + O(H^4)}$$

$$\begin{aligned} m(\tau, H) &= -\sqrt{2}\tau a(\tau) + H^2\,b(\tau) + O(H^4) \\ h(\tau, H) &= C_h H\left[c(\tau) + H^2\,d(\tau) + O(H^4)\right] \end{aligned}$$

$$\Phi(\eta) = -\frac{\eta^2}{8\pi}\log\eta^2 + \sum_{k=0}^{\infty}\Phi_k\eta^k$$

$$\begin{aligned} \Phi_{low}(\eta) &= \tilde{G}_1\eta^{\frac{1}{8}} + \tilde{G}_2\eta^{-\frac{7}{4}} + \tilde{G}_3\eta^{-\frac{29}{8}} + \ldots && \text{for real} \quad \eta \to +\infty \\ \Phi_{high}(\eta) &= G_2(-\eta)^{-\frac{7}{4}} + G_4(-\eta)^{-\frac{22}{4}} + G_6(-\eta)^{-\frac{37}{4}} + \ldots && \text{for real} \quad \eta \to -\infty \end{aligned}$$

Lattice calculations

Onsager's solution:

$$F(\tau,0)=\log\sqrt{2}\cosh(2\beta)+\int_0^\pi\frac{d\theta}{2\pi}\log\left[1+\left(1-\frac{\cos^2\theta}{1+\tau^2}\right)^{1/2}\right]$$

$$a(\tau)=1-\frac{3\tau^2}{16}+\frac{137\tau^4}{1536}+O(\tau^6)$$

$$A(\tau)=-\frac{2\mathcal{G}}{\pi}-\frac{\log 2}{2}+\frac{\tau}{2}-\frac{\tau^2(1+5\log 2)}{4\pi}-\frac{\tau^3}{12}+\frac{5\tau^4(1+6\log 2)}{64\pi}+O(\tau^5)$$

Magnetization:

$$M(\tau,0)=(1-k(\tau)^2)^{1/8},\quad \tau<0,\quad k=k(\tau)=(\sqrt{1+\tau^2}+\tau)^2$$

$$c(\tau)=1+\frac{\tau}{4}+\frac{15\tau^2}{128}-\frac{9\tau^3}{512}-\frac{4333\tau^4}{98304}+O(\tau^5)$$

Susceptibility

Susceptibility (Orrick, Nickel, Guttmann, Perk (2001)):

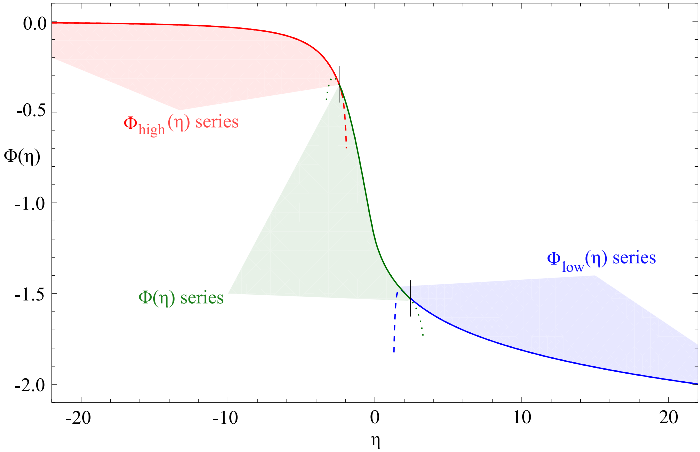
$$\chi(\tau)_{\text{ONGP}} = -2^{-\frac{7}{8}} C_h^2 G''(0) |\tau|^{-\frac{7}{4}} \left(1 + \frac{\tau}{2} + \frac{5\tau^2}{8} + \frac{3\tau^3}{16} - \frac{23\tau^4}{384} + O(\tau^5)\right) + e(\tau) + f(\tau) \log|\tau| + O(\tau^3 \log|\tau|)$$

$$\chi(\tau) = -2^{-\frac{7}{8}} C_h^2 G''(0) |\tau|^{-\frac{7}{4}} a(\tau)^{-\frac{7}{4}} c(\tau)^2 - \frac{\partial^2 F_{sub}(\tau, H)}{\partial H^2}\Big|_{H=0} - 2B(\tau) + \frac{\tau\, a(\tau)\, b(\tau)}{\sqrt{2}\,\pi} \left(1 + \log\left(2\tau^2 a(\tau)\right)\right)$$

$$B(\tau) = 0.0520666225469 + 0.0769120341893\,\tau + 0.0360200462309\,\tau^2 + O(\tau^3)$$

$$b(\tau) = \mu_h \left(1 + \frac{\tau}{2} + O(\tau^2)\right), \qquad \mu_h = 0.071868670814$$

$$\left(2^{-\frac{7}{8}} C_h^2 G''(0)\right)^{-1} \frac{\partial^2 F_{sub}(\tau, H)}{\partial H^2}\Big|_{H=0} = -\frac{1}{384} |\tau|^{\frac{9}{4}} + \dots$$



Scaling Function for the 2D Ising model in a magnetic field

Data for the function $\Phi(\eta)$

	CTM (This work)	TFFSA	Ext. DR	Other
Φ_0	$-1.197733383797993(1)$	-1.1977331	-1.1977320	$-1.197733383797993..$
Φ_1	$-0.318810124891(1)$	-0.3188103	-0.3188192	$-0.3188101248906...$
Φ_2	$0.110886196683(2)$	0.1108867	0.1108915	—
Φ_3	$0.01642689465(2)$	0.0164266	0.0164252	—
Φ_4	$-2.639978(1) \times 10^{-4}$	-2.64×10^{-4}	-2.64×10^{-4}	—
Φ_5	$-5.140526(1) \times 10^{-4}$	-5.14×10^{-4}	-5.14×10^{-4}	—
Φ_6	$2.08865(1) \times 10^{-4}$	2.07×10^{-4}	2.09×10^{-4}	—
Φ_7	$-4.4819(1) \times 10^{-5}$	-4.52×10^{-5}	-4.48×10^{-5}	—

Data for the function $\Phi(\eta)$

	CTM (This work)	Low-T DR	From References
$\tilde{G}_1$	$-1.3578383417066(1)$	-1.35783835	$-1.357838341706595...$ MW
$\tilde{G}_2$	$-0.048953289720(1)$	-0.0489589	$-0.0489532897203...$ BMW,TM,ONGP
$\tilde{G}_3$	$0.038863932(3)$	0.0388954	0.0387529 ;MW $0.039(1)$ ZLF
$\tilde{G}_4$	$-0.068362119(2)$	-0.0685060	-0.0685535 MW; $-0.0685(2)$ ZLF
$\tilde{G}_5$	$0.18388370(1)$	0.18453	—
$\tilde{G}_6$	$-0.6591714(1)$	-0.66215	—
$\tilde{G}_7$	$2.937665(3)$	2.952	—
$\tilde{G}_8$	$-15.61(1)$	-15.69	—

	CTM (This work)	High-T DR (FZ)	From References	
G_2	$-1.8452280782328(2)$	-1.8452283	$-1.845228078232838...$	(BMW,TM)
G_4	$8.333711750(5)$	8.33410	$8.33370(1)$	(CHPV)
G_6	$-95.16896(1)$	-95.1884	$-95.1689(4)$	(CHPV)
G_8	$1457.62(3)$	1458.21	$1457.55(11)$	(CHPV)
G_{10}	$-25891(2)$	-25889	$-25884(13)$	(CHPV)

Conclusion

- The numerical corner-transfer matrix algorithm demonstrates remarkable power for the 2D lattice Ising model.
- Among other results we showed excellent agreement with the field theory predictions by Fonseca & Zamolodchikov for the scaling function.
- The CTM method can be naturally formulated for any statistical lattice system including vertex models in 2D and 3D, and thus (1+1)D and (2+1)D quantum systems.
- We have implemented fast parallelized codes running on various computers. For example, CPU time for the Ising model calculations was about 9000 hours (1 CPU equivalent) with parallelization level of 15-50 CPU's.
- We aim to apply it to a range of statistical mechanics problems – self-avoiding polygons, 3D Ising model, etc.