High-fugacity expansion and crystalline ordering for non-sliding hard-core lattice particle systems

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Abstract

We establish existence of order-disorder phase transitions for a class of "non-sliding" hard-core lattice particle systems on a lattice in two or more dimensions. All particles have the same shape and can be made to cover the lattice perfectly in a finite number of ways. We also show that the pressure and correlation functions have a convergent expansion in powers of the inverse of the fugacity. This implies that the Lee-Yang zeros lie in an annulus with finite positive radii.

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1. Introduction

One of the most interesting open problems in the theory of equilibrium statistical mechanics, is to prove the existence of order-disorder phase transitions in continuum particle systems. While such fluid-crystal transitions are ubiquitous in real systems and are observed in computer simulations of systems with effective pair potentials, there are no proofs, or even good heuristics, for showing this mathematically. A paradigmatic example of this phenomenon is the fluid-crystal transition for hard spheres in 3 dimensions, observed in simulations and experiments [WJ57, AW57, PM86, IK15]. Whereas, in 2 dimensions, crystalline states are ruled out by the Mermin-Wagner theorem [Ri07], it is believed that there are other transitions for hard discs [BK11] (see [St88] or [Mc10, section 8.2.3] for a review), though none have, as of yet, been proven. Such transitions are purely geometric. They are driven by entropy and depend only on the density, that is, on the volume fraction taken up by the hard particles.

The situation is different for lattice systems, where there are many examples for which such entropy-driven transitions have been proven. A simple example is that of hard "diamonds" on the square lattice (see figure 1.1*a*), which is a model on $\mathbb{Z}^2$ with nearest-neighbor exclusion. As was shown by Dobrushin [Do68], this model transitions from a low-density disordered state to a high-density crystalline phase, where the even or odd sublattice is preferentially occupied. The heuristics of this transition had been understood earlier (the hard diamond model related to the 0-temperature limit of the antiferromagnetic Ising model for which the exponential of the magnetic field plays the role of the fugacity [BK73, LRS12]), for instance by Gaunt and Fisher [GF65], who extrapolated a low- and high-fugacity expansion of the pressure $p(z)$ to find a singularity at a critical fugacity $z_c > 0$. A similar analysis was carried out for the nearest neighbor exclusion on $\mathbb{Z}^3$ by Gaunt [Ga67].

The low-fugacity expansion in powers of the fugacity z dates back to Ursell [Ur27] and Mayer [Ma37]. Its radius of convergence was bounded below by Groeneveld [Gr62] for positive pair-potentials and by Ruelle [Ru63] and Penrose [Pe63] for general pair-potentials.

The high-fugacity expansion is an expansion in powers of the inverse fugacity $y \equiv z^{-1}$. As far as we know, it was first considered by Gaunt and Fisher [GF65] for the hard diamond model, without any indication of its having a positive radius of convergence.

In this paper we prove, using an extension of Pirogov-Sinai theory [PS75, KP84], that the high-fugacity expansion has a positive radius of convergence for a class of hard-core lattice particle systems in $d \geqslant 2$ dimensions. We call these *non-sliding* models. In addition, we show that these systems exhibit high-density crystalline phases, which, combined with the convergence of the low-fugacity expansion proved in [Gr62, Ru63, Pe63], proves the existence of an order-disorder phase transition for these models. A preliminary account of this work, without proofs, is in [JL17].

Non-sliding models are systems of identical hard particles which have a finite number of maximal density perfect coverings of the infinite lattice, and are such that any defect in a covering leaves an amount of empty space that is proportional to its size, and that a particle configuration is characterized by its defects (this will be made precise in the following). This class includes many of the models for which crystallization has been proved, namely the hard diamond (see figure 1.1*a*) model discussed above, as well as the hard cross model (see figure 1.1*b*), which corresponds to the third-nearest-neighbor exclusion on $\mathbb{Z}^2$, and the hard hexagon model on the triangular lattice (see figure 1.1*c*), which corresponds to the nearest-neighbor exclusion on the triangular lattice.

The hard cross model was studied by Heilmann and Præstgaard [HP74], who gave a sketch of a proof that it has a crystalline high-density phase. Eisenberg and Baram [EB05] computed the first 6 terms of the high-fugacity expansion for this model, and conjectured that it should have a *first-order* order-disorder phase transition. We will prove the convergence of the high-fugacity expansion, and reproduce Heilmann and Præstgaard's result, but will stop short of proving the order of the phase transition, for which new techniques would need to be developed. We will also extend this result to the hard cross model on a fine lattice, although the present techniques do not allow us to go to the continuum.

The hard hexagon model on the triangular lattice was shown to be exactly solvable by Baxter - [Ba80, Ba82], and to be crystalline at high densities. The exact solution provides an (implicit) expression for the pressure $p(z)$, from which the high-fugacity expansion can be obtained, as shown by Joyce [Jo88].

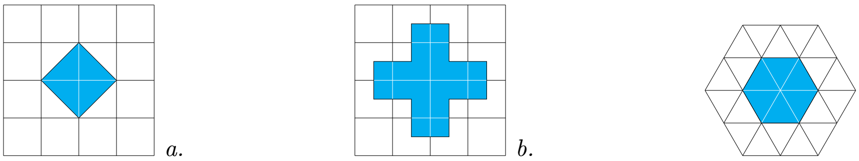

fig 1.1: Three non-sliding hard-core lattice particle systems.
a. The hard diamond model is equivalent to the nearest neighbor exclusion on $\mathbb{Z}^2$.
b. The hard cross model is equivalent to the third-nearest neighbor exclusion on $\mathbb{Z}^2$.
c. The hard hexagon model is equivalent to the nearest neighbor exclusion on the triangular lattice.

1.1. Non-sliding hard-core lattice particle models

Consider a d-dimensional lattice Λ_∞, which we consider as a graph, that is, every vertex of Λ_∞ has a set of *neighbors*. We denote the graph distance on Λ_∞ by Δ, in terms of which $x, x' \in \Lambda_\infty$ are neighbors if and only if $\Delta(x, x') = 1$. We will consider systems of identical particles on Λ_∞ with hard core interactions. We will represent the latter by assigning a *support* to each particle, which is a connected and bounded subset $\omega \subset \mathbb{R}^d$, and forbid the supports of different particles from overlapping. In the examples mentioned above, the shapes would be a diamond, a cross or a hexagon (see figure 1.1). We define the grand-canonical partition function of the system at activity $z > 0$ on any bounded $\Lambda \subset \Lambda_\infty$ as

$$\Xi_\Lambda(z) = \sum_{X \subset \Lambda} z^{|X|} \prod_{x \neq x' \in X} \varphi(x, x') \tag{1.1}$$

in which X is a particle configuration in Λ (that is, a set of lattice points $x \in \Lambda$ on which particles are located), $|X|$ is the cardinality of X, and, denoting $\omega_x := \{x + y,\ y \in \omega\}$, $\varphi(x, x') \in \{0, 1\}$ enforces the hard core repulsion: it is equal to 1 if and only if $\omega_x \cap \omega_{x'} = \emptyset$. In the following, a subset $X \subset \Lambda_\infty$ is said to be a *particle configuration* if $\varphi(x, x') = 1$ for every $x \neq x' \in X$, and we denote the set of particle configurations in Λ by $\Omega(\Lambda)$. Note that the sum over X is a finite sum, since the hard-core repulsion imposes a bound on $|X|$:

$$|X| \leqslant N_{\max}. \tag{1.2}$$

In addition, note that several different shapes can, in some cases, give rise to the same partition function. For example, the hard diamond model is equivalent to a system of hard disks of radius $r \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$.

Our main result concerns hard-core lattice particles that satisfy the *non-sliding* property, which, roughly, means that the system admits only a finite number of perfect coverings, that any defect in a covering induces an amount of empty space that is proportional to its volume, and that any particle configuration is entirely determined by its defects. More precisely, defining σ_x as the set of lattice sites that are covered by a particle at x:

$$\sigma_x := \omega_x \cap \Lambda_\infty \tag{1.3}$$

given a particle configuration $X \in \Omega(\Lambda)$, we define the set of *empty* sites as those that are not covered by any particle:

$$\mathcal{E}_\Lambda(X) := \{y \in \Lambda, \quad \forall x \in X,\ y \notin \sigma_x\} \tag{1.4}$$

A *perfect covering* is defined as a particle configuration $X \in \Omega(\Lambda_\infty)$ that leaves no empty sites: $\mathcal{E}_{\Lambda_\infty}(X) = \emptyset$.

Definition 1.1

(*sliding*)

A hard-core lattice particle system, as defined above, is said to be *non-sliding* if the following hold.

- There exists $\tau > 1$, a *periodic* perfect covering $\mathcal{L}_1$, and a finite family $(f_2, \cdots, f_\tau)$ of isometries of Λ_∞ such that, for every i, $\mathcal{L}_i \equiv f_i(\mathcal{L}_1)$ is a perfect covering (see figure 2.2 for an example).
- Given a bounded *connected* particle configuration $X \in \Omega(\Lambda_\infty)$ (that is, a configuration in which the union of particle supports $\bigcup_{x\in X} \sigma_x$ is connected), we define $\mathbb{S}(X)$ as the set of particle configurations X' that contain X and *isolate* it from the rest of Λ_∞ without leaving any empty space (see figures 2.5 and 2.6):

$$\mathbb{S}(X) := \{X' \in \Omega(\Lambda_\infty),\ X' \supset X,\ \forall x \in X,\ \forall x' \in X',\ \Delta(\mathcal{E}_{\Lambda_\infty}(X'), \sigma_x) > 1,\ \Delta(\sigma_x, \sigma_{x'}) \leqslant 1\} \tag{1.5}$$

  in which, we recall, Δ denotes the graph distance on Λ_∞. In order to be non-sliding, a model must be such that, for every bounded connected X, $\mathbb{S}(X) = \emptyset$, or, $\forall X' \in \mathbb{S}(X)$, there exists a unique $\mu \in \{1, \cdots, \tau\}$ such that $X' \subset \mathcal{L}_\mu$.

Remark: In non-sliding models, every defect induces an amount of empty space proportional to its size, because any connected particle configuration X that is not a subset of any perfect covering must have $\mathbb{S}(X) = \emptyset$, which means that there must be some empty space next to it. In addition, a particle configuration is determined by the empty space and the particles surrounding it, since the remainder of the particle configuration consists of disconnected groups, each of which is the subsets of a perfect covering. The position of the particles surrounding it determines uniquely which one of the perfect coverings it is a subset of.

Our main result is that the finite-volume *pressure* of non-sliding hard-core particles systems, defined as

$$p_\Lambda(z) := \frac{1}{|\Lambda|} \log \Xi_\Lambda(z) \tag{1.6}$$

satisfies

$$p(z) := \lim_{\Lambda \to \Lambda_\infty} p_\Lambda(z) = \rho_m \log z + f(y) \tag{1.7}$$

in which ρ_m is the maximal density $\rho_m = \lim_{\Lambda\to\Lambda_\infty} \frac{N_{\max}}{|\Lambda|}$ and f is an analytic function of $y \equiv \frac{1}{z}$ for small values of y. The expansion of f in powers of y is called the *high-fugacity expansion* of the system. Note that the infinite-volume pressure $p(z)$ does not depend on the boundary conditions [Ru99].

1.2. Low-fugacity expansion

The main ideas underlying the high-fugacity expansion come from the low-fugacity expansion, which we will now briefly review. It is an expansion of p_Λ in powers of the fugacity z, and its formal derivation is fairly straightforward: defining the *canonical* partition function as

$$Z_\Lambda(k) := \sum_{\substack{X\subset\Lambda\\ |X|=k}} \prod_{x\neq x'\in X} \varphi(x,x') \tag{1.8}$$

as the number of particle configurations with k particles, (1.1) can be rewritten as

$$\Xi_\Lambda(z) = \sum_{k=0}^{N_{\max}} z^k Z_\Lambda(k). \tag{1.9}$$

Injecting (1.9) into (1.6), we find that, formally,

$$p_\Lambda(z) = \sum_{k=1}^{\infty} z^k b_k(\Lambda) \tag{1.10}$$

with

$$b_k(\Lambda) := \frac{1}{|\Lambda|}\sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{\substack{k_1,\cdots,k_n\geqslant 1\\ k_1+\cdots+k_n=k}} Z_\Lambda(k_1)\cdots Z_\Lambda(k_n). \tag{1.11}$$

As was shown in [Gr62, Ru63, Pe63], there is a remarkable cancellation that eliminates the terms in $b_k(\Lambda)$ that diverge as $\Lambda\to\Lambda_\infty$, so that $b_k(\Lambda)\to b_k$ when $\Lambda\to\Lambda_\infty$. This becomes obvious when the $b_k(\Lambda)$ are expressed as integrals over Mayer graphs. In addition, the radius of convergence $R(\Lambda)$ of (1.10) converges to $R>0$, which is at least as large as the radius of convergence of $\sum_{k=1}^\infty b_k z^k$.

1.3. High-fugacity expansion

The low-fugacity expansion is obtained by perturbing around the vacuum state by adding particles to it. The high-fugacity expansion will be obtained by perturbing perfect coverings by introducing *defects*. Single-particle defects come with a cost $y\equiv z^{-1}$, which is, effectively, the fugacity of a hole. The main idea, due to Gaunt and Fisher [GF65], is to carry out a low-activity expansion for the defects, which is similar to the low-fugacity expansion described above. Let us go into some more detail in the example of the hard diamond model.

We will take Λ to be a $2n\times 2n$ torus, which can be completely packed with diamonds (see figure 2.1). The number of perfect covering configurations is

$$\tau = 2 \tag{1.12}$$

and the maximal number of particles and maximal density are

$$N_{\max} = \rho_m|\Lambda|, \quad \rho_m = \frac{1}{2}. \tag{1.13}$$

We denote the number of configurations that are *missing* k particles as

$$Q_\Lambda(k) := Z_\Lambda(N_{\max}-k) \tag{1.14}$$

in terms of which

$$\Xi_\Lambda(z) = \tau z^{N_{\max}} \sum_{k=0}^{N_{\max}} \left(\frac{1}{\tau} z^{-k} Q_\Lambda(k)\right) \tag{1.15}$$

(we factor τ out because $Q_\Lambda(0) = \tau$ and we wish to expand the logarithm in (1.6) around 1). We thus have, formally

$$p_\Lambda(y) = \frac{1}{|\Lambda|}\log\tau + \rho_m \log z + \sum_{k=1}^{N_{\max}} y^k c_k(\Lambda) \tag{1.16}$$

where $y \equiv z^{-1}$ and

$$c_k(\Lambda) := \frac{1}{|\Lambda|}\sum_{n=1}^{k} \frac{(-1)^{n+1}}{n\tau^n} \sum_{\substack{k_1,\cdots,k_n \geqslant 1 \\ k_1+\cdots+k_n = k}} Q_\Lambda(k_1)\cdots Q_\Lambda(k_n). \tag{1.17}$$

The first 9 $c_k(\Lambda)$ are reported in [GF65, table XIII] and, as for the low-fugacity expansion, there is a remarkable cancellation that ensures that these coefficients converge to c_k as $\Lambda \to \Lambda_\infty$. *But*, unlike the low-fugacity expansion, there is no *systematic* way of exhibiting this cancellation for general hard-core lattice particle systems. In fact there are many example of systems in which $c_2(\Lambda)$ diverges as $\Lambda \to \Lambda_\infty$, like the nearest-neighbor exclusion model in 1 dimension (which maps, exactly, to the 1-dimensional monomer-dimer model), for which

$$Q_\Lambda(2) = \frac{1}{192}(|\Lambda|^2 - 4)|\Lambda|^2, \quad Q_\Lambda(1) = \frac{1}{4}|\Lambda|^2, \quad c_1(\Lambda) = \frac{1}{8}|\Lambda|, \quad c_2(\Lambda) = -\frac{1}{192}|\Lambda|(5|\Lambda|^2 + 4). \tag{1.18}$$

Note that the pressure for this model is

$$p(y) - \rho_m \log z = \log\left(\frac{1+\sqrt{1+4z}}{2}\right) - \frac{1}{2}\log z = \log\left(\sqrt{1+\frac{1}{4}y} + \frac{1}{2}\sqrt{y}\right) \tag{1.19}$$

which is not an analytic function of $y \equiv z^{-1}$ at $y = 0$ (though it is an analytic function of $\sqrt{y}$). Clearly, this model does not satisfy the non-sliding property.

Our approach, in this paper, is to prove that, for non-sliding models, the pressure is analytic in a disk around $y = 0$, thus proving the validity of the Gaunt-Fisher expansion for non-sliding systems.

Remark: Before moving to our main result, let us note that, at finite temperature, lattice gases of particles with a *bounded* pair potential φ that admit a convergent low-fugacity expansion (for example for summable potentials) also admit a high-fugacity expansion. This follows immediately from the Kramers-Wannier [KW41] duality of the corresponding Ising model, which implies that

$$p_\Lambda(z) = \log(ze^{-\frac{1}{2}\alpha})p(ye^\alpha), \quad e^\alpha := e^{\beta\sum_{x\in\Lambda}\varphi(|x|)} \tag{1.20}$$

The radius of convergence R_y of the expansion in y is therefore related to the radius R_z of convergence of the expansion in z: $R_y = R_z e^{-\alpha}$. This coincides, at sufficiently high temperature, with the results of Gallavotti, Miracle-Sole and Robinson [GMR67], who prove analyticity for small values of $\frac{z}{1+z}$.

1.4. High-fugacity expansion and Lee-Yang zeros

As was first remarked by Lee and Yang [YL52, LY52], a powerful tool to study the thermodynamic properties of a system is to compute the positions of the roots of the partition function as a function of the fugacity z, called the *Lee-Yang zeros* of the model. In particular, the logarithm of the partition function and, consequently, the pressure, diverge at the Lee-Yang zeros, so when such roots lie on the positive real axis, they signal the presence of a phase transitions. Let us denote the set of Lee-Yang zeros of a hard-core lattice particle system by $\{\xi_1(\Lambda), \cdots, \xi_{N_{\max}}(\Lambda)\}$. The convergence of the low-fugacity expansion within its radius of convergence $R_z > 0$ implies that every Lee-Yang zero satisfies $|\xi_i(\Lambda)| \geqslant R_z$, and that this inequality is sharp. Similarly, when the high-fugacity expansion has a positive radius of convergence $R_y > 0$, every Lee-Yang zero must satisfy

$$R_z \leqslant |\xi_i(\Lambda)| \leqslant R_y^{-1} \tag{1.21}$$

and these inequalities are sharp. In addition, writing the partition function as

$$\Xi_\Lambda(z) = \prod_{i=1}^{N_{\max}} \left(1 - \frac{z}{\xi_i(\Lambda)}\right) = \frac{z^{N_{\max}}}{\prod_{i=1}^{N_{\max}}(-\xi_i(\Lambda))} \prod_{i=1}^{N_{\max}} (1 - y\xi_i(\Lambda)) \tag{1.22}$$

we rewrite the high-fugacity expansion (1.16) as

$$p_\Lambda(y) = \rho_m \log z - \frac{1}{|\Lambda|} \sum_{i=1}^{N_{\max}} \log(-\xi_i(\Lambda)) - \sum_{k=1}^{\infty} \frac{y^k}{k} \left(\frac{1}{|\Lambda|} \sum_{i=1}^{N_{\max}} \xi_i^k(\Lambda)\right) \tag{1.23}$$

which, in particular, implies that

$$\prod_{i=1}^{N_{\max}} (-\xi_i(\Lambda)) = \frac{1}{Q_\Lambda(0)}, \quad c_k(\Lambda) = -\frac{1}{k}\left(\frac{1}{|\Lambda|} \sum_{i=1}^{N_{\max}} \xi_i^k(\Lambda)\right). \tag{1.24}$$

When taking the thermodynamic limit, c_k is proportional to the average of the k-th power of ξ weighted against the limiting distribution of Lee-Yang zeros. Thus, the high-fugacity expansion converges if and only if the average of ξ^k divided by k grows at most exponentially in k.

Remark: For bounded potentials, using the Kramers-Wannier duality in (1.20), we find that the Lee-Yang zeros all lie in an annulus of radii R_z and e^α/R_z. Note that if one were to consider a hard-core model as the limit of a bounded repulsive potential, the hard-core limit would correspond to taking $\alpha \to \infty$. This implies that some zeros move out to infinity. This does not, however, imply that in the hard-core limit $\Xi_\Lambda(y)$ vanishes for $y = 0$: indeed the distribution of Lee-Yang zeros does not approach the hard-core limit continuously, as is made obvious by the fact that the number of Lee-Yang zeros for finite potentials is $|\Lambda|$, whereas it is $N_{\max}$ in the hard-core limit.

1.5. Main result

Our main result concerns the partition function and correlation functions of non-sliding hard-core lattice particle systems, with the following boundary conditions. First of all, we require the set Λ to be *tiled*, by which we mean that there must exist $\mu \in \{1, \cdots, \tau\}$ and a set $S \subset \Lambda_\infty$ such that

$$\Lambda = \bigcup_{x \in S} \sigma_x, \quad \sigma_x \cap \sigma_{x'} = \emptyset, \ \forall x \neq x' \in S. \tag{1.25}$$

For $\nu \in \{1, \cdots, \tau\}$, we define Ω_ν as the set of particle configurations such that

- if the complement of Λ were covered by a ν-covering, then the particles in Λ would not overlap with those outside Λ,
- the space left empty by the configuration must not neighbor the boundary of Λ:

$$\Omega_\nu(\Lambda) := \{X \subset \Lambda, \quad \forall x \in \mathcal{L}_\nu \setminus \Lambda, \ \forall x' \in X, \quad \varphi(x, x') = 1, \quad \Delta(\mathcal{E}_\Lambda(X), \sigma_x) > 1\} \tag{1.26}$$

in which, we recall, Δ denotes the graph distance on Λ_∞. We chose this particular boundary condition in order to make the discussion below simpler. More natural boundary conditions would presumably be treatable as well, though they might complicate the proof somewhat. In addition, we generalize the notion of fugacity by allowing it to depend on the position of the particle:

given a function $\underline{z}:\Lambda_\infty\to[0,\infty)$, we define the partition function with fugacity $\underline{z}$ and boundary condition ν as

$$\Xi_\Lambda^{(\nu)}(\underline{z})=\sum_{X\in\Omega_\nu(\Lambda)}\left(\prod_{x\in X}\underline{z}(x)\right)\prod_{x\neq x'\in X}\varphi(x,x'). \tag{1.27}$$

Since the infinite-volume pressure is independent of the boundary condition, it can be recovered from $\Xi_\Lambda^{(\nu)}(\underline{z})$ by setting $\underline{z}(x)\equiv z$. By allowing the fugacity to depend on the position of the particle, we can compute the $\mathfrak{n}$-*point truncated correlation functions* of the system with ν-boundary conditions at fugacity z, defined as

$$\rho_{n,\Lambda}^{(\nu)}(\mathfrak{x}_1,\cdots,\mathfrak{x}_n):=\left.\frac{\partial^{\mathfrak{n}}}{\partial\log\underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\log\Xi_\Lambda^{(\nu)}(\underline{z})\right|_{\underline{z}(x)\equiv z} \tag{1.28}$$

as well as its infinite-volume limit

$$\rho_n^{(\nu)}(\mathfrak{x}_1,\cdots,\mathfrak{x}_n):=\lim_{\Lambda\to\Lambda_\infty}\rho_{n,\Lambda}^{(\nu)}(\mathfrak{x}_1,\cdots,\mathfrak{x}_n). \tag{1.29}$$

Note that the 1-point correlation function is the local density.

Our main result is summarized in the following theorem.

Theorem 1.2

(*crystallization and high-fugacity expansion*)

Consider a non-sliding hard-core lattice particle system and a boundary condition $\nu\in\{1,\cdots,\tau\}$. For any $\mathfrak{n}\geqslant 1$ and $\mathfrak{x}_1,\cdots,\mathfrak{x}_{\mathfrak{n}}\in\Lambda_\infty$, both $p(z)-\rho_m\log z$ and the n-point truncated correlation function $\rho_n^{(\nu)}(\mathfrak{x}_1,\cdots,\mathfrak{x}_n)$ are analytic functions of $y\equiv z^{-1}$, with a positive radius of convergence y_0.

Furthermore, defining the average density as

$$\rho:=\lim_{\Lambda\to\Lambda_\infty}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\rho_{1,\Lambda}^{(\nu)}(x) \tag{1.30}$$

both $p+\rho_m\log(\rho_m-\rho)$ and $\rho_n^{(\nu)}(\mathfrak{x}_1,\cdots,\mathfrak{x}_n)$ are analytic functions of $\rho_m-\rho$, with a positive radius of convergence.

In addition, if $|y|$ is sufficiently small, then the particles are much more likely to be on the $\mathcal{L}_\nu$ sublattice than the others: for every $x\in\mathcal{L}_\nu$ and $x'\in\Lambda_\infty\setminus\mathcal{L}_\nu$,

$$\rho_1^{(\nu)}(x)=\rho_m+O(y),\quad \rho_1^{(\nu)}(x')=O(y). \tag{1.31}$$

Remark: We show that the analyticity of the pressure in y implies analyticity in $\rho_m-\rho$. The converse is not necessarily true. In particular, if $p-\rho_m\log z$ is analytic in y^α for some α (as is the case for the 1-dimensional nearest neighbor exclusion, for which $\alpha=\frac{1}{2}$), then it is also analytic in $\rho_m-\rho$.

2. Non-sliding hard-core lattice particle models

In this section, we present several examples of non-sliding hard-core lattice particle models.

1 - Let us start with the hard diamond model, or rather, a generalization to the "hyperdiamond" model $d \geqslant 2$-dimensions, which is equivalent to the nearest neighbor exclusion on $\mathbb{Z}^d$. It is formally defined by specifying the lattice $\Lambda_\infty = \mathbb{Z}^d$ and the hyperdiamond shape $\omega \subset \mathbb{R}^2$ (see figure 1.1*a*):

$$\omega = \left\{(x_1, \cdots, x_d) \in (-1,1)^d,\ \textstyle\sum_{i=1}^n |x_i| < 1\right\} \cup \{(0, \cdots, 0, 1)\right\}. \tag{2.1}$$

Note the adjunction of the point $(0, \cdots, 0, 1)$, whose absence would prevent the existence of any perfect covering (see figure 2.1). The notion of *connectedness* in Λ_∞ is defined as follows: two points are connected if and only if they are at distance 1 from each other. There are 2 perfect coverings (see figure 2.1):

$$\mathcal{L}_1 = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d,\ x_1 + \cdots + x_d \text{ even}\}, \quad \mathcal{L}_2 = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d,\ x_1 + \cdots + x_d \text{ odd}\} \tag{2.2}$$

which are related to each other by the translation by $(0, \cdots, 0, 1)$. Finally, this model satisfies the non-sliding condition because any pair $x_1, x_2 \in \mathbb{Z}^d$ of hyperdiamonds whose supports are disjoint and connected are both in the same sublattice: $(x_1, x_2) \in \mathcal{L}_1^2 \cup \mathcal{L}_2^2$, and the distinct sublattices do not overlap $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$. Connected hyperdiamond configurations are, therefore, always subsets of $\mathcal{L}_1$ or of $\mathcal{L}_2$, and one can find which one it is by from the position of a single one of its particles.

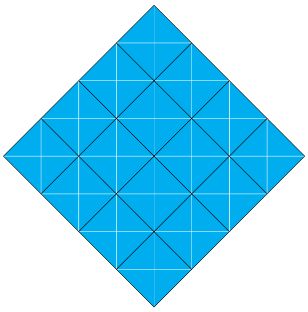

fig 2.1: Perfect covering of diamonds. There are 2 inequivalent such coverings, obtained by translating the one depicted here.

2 - Let us now consider the hard-cross model (see figure 1.1*b*), for which $\Lambda_\infty = \mathbb{Z}^2$, and

$$\omega = \left\{(n_x + x, n_y + y),\ (x, y) \in (-\tfrac{1}{2}, \tfrac{1}{2})^2,\ (n_x, n_y) \in \{-1, 0, 1\}^2,\ |n_x| + |n_y| \leqslant 1\right\}. \tag{2.3}$$

There are 10 perfect coverings (see figure 2.2):

$$\mathcal{L}_1 = \{(n_x + 2n_y, 2n_x - n_y),\ (n_x, n_y) \in \mathbb{Z}^2\}, \quad \mathcal{L}_2 = \{(-n_x + 2n_y, 2n_x + n_y),\ (n_x, n_y) \in \mathbb{Z}^2\} \tag{2.4}$$

and, for $p \in \{2, 3, 4, 5\}$,

$$\mathcal{L}_{2p-1} = v_p + \mathcal{L}_1, \quad \mathcal{L}_{2p} = v_p + \mathcal{L}_2 \tag{2.5}$$

with $v_2 = (1, 0)$, $v_3 = (0, 1)$, $v_4 = (-1, 0)$ and $v_5 = (0, -1)$. The $\mathcal{L}_{2p-1}$ are related to $\mathcal{L}_1$ by translations, as are the $\mathcal{L}_{2p}$ related to $\mathcal{L}_2$, and $\mathcal{L}_2$ is mapped to $\mathcal{L}_1$ by the vertical reflection. Let us now check the non-sliding property. We first introduce the following definitions: two crosses at x, x' whose supports are connected and disjoint are said to be (see figure 2.3)

- *left-packed* if $x - x' \in \{(1, 2), (-2, 1), (-1, -2), (2, -1)\}$
- *right-packed* if $x - x' \in \{(2, 1), (-1, 2), (-2, -1), (1, -2)\}$

- *stacked* if $x - x' \in \{(3,0),(0,3),(-3,0),(0,-3)\}$.

Now, consider a connected configuration of crosses X.

- If $|X| = 1$, then $\mathbb{S}(X)$ (see definition 1.1) consists of the two configurations in figure 2.5, each of which is the subset of a unique sublattice $\mathcal{L}_\mu$.
- If X contains at least one pair $x, x' \in X$ of stacked crosses, which, without loss of generality, we assume satisfies $x - x' = (-3,0)$, then one of the two sites $x + (1,1)$ or $x + (2,1)$ cannot be covered by any other cross (see figure 2.4a), which implies that $\mathbb{S}(X) = \emptyset$.
- If every pair of crosses in X is either left- or right-packed,and there exists at least one triplet $x, x', x'' \in X$ whose supports are connected and disjoint, and is such that x, x' is right-packed and x, x'' is left-packed. Without loss of generality, we assume that $x - x' = (2,1)$ and $x - x'' = (-1,-2)$ (see figure 2.4b) or $x - x'' = (-2,1)$ (see figure 2.4c). In the former case, the site $x + (-1,1)$ cannot be covered by any other crosses. In the latter case, one of the three sites $x + (-1,-2)$, $x + (0,-2)$ or $x + (1,-2)$ cannot be covered by any other cross. Thus, $\mathbb{S}(X) = \emptyset$.
- Finally, suppose that every pair of crosses is left-packed (the case in which they are all right-packed is treated identically). Let Y be a pair of left-packed crosses, $\mathbb{S}(Y)$ consists of a single configuration, depicted in figure 2.6, which is a subset of a unique sublattice $\mathcal{L}_\mu$. Since there is a unique way of isolating each left-packed pair in X, there is a single way of isolating X, that is, $\mathbb{S}(X)$ consists of a single configuration, which is the union over left-packed pairs Y in X of the unique configuration in $\mathbb{S}(Y)$, and is, therefore, a subset of a unique sublattice $\mathcal{L}_\mu$.

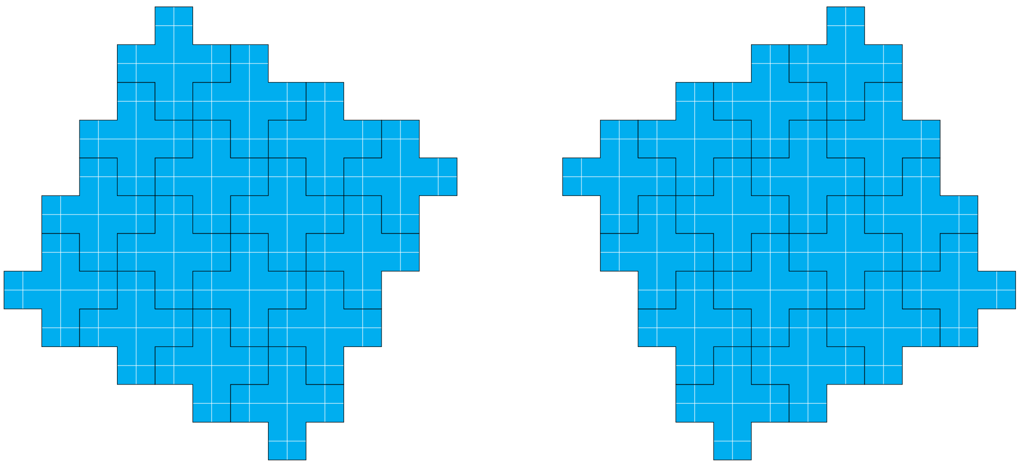

fig 2.2: Perfect coverings of crosses. There are 10 inequivalent such coverings, obtained by translating each of the ones depicted here in 5 inequivalent ways. These two coverings are related to each other by a reflection.

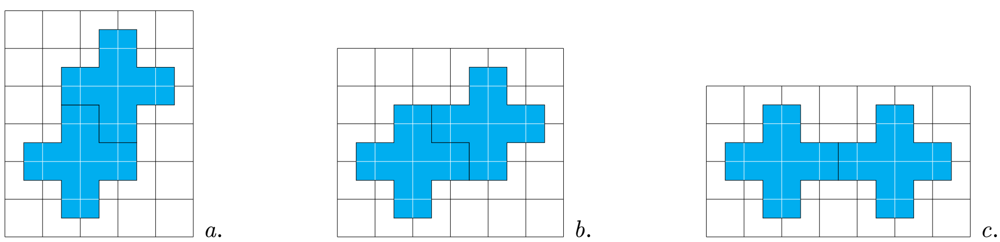

fig 2.3: Pairs of crosses that are (a) left-packed, (b) right-packed and (c) stacked.

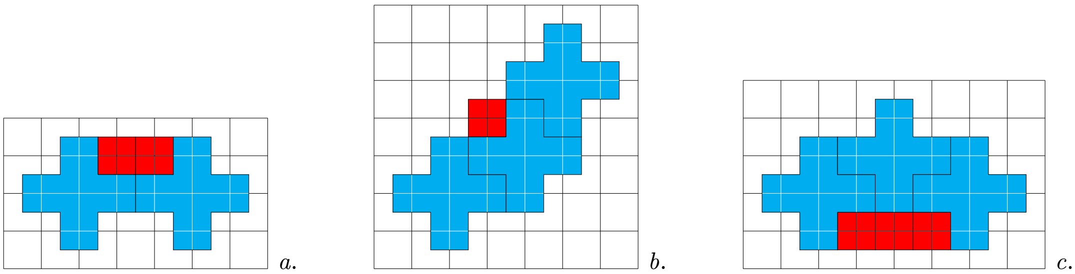

fig 2.4: Connected configurations that cannot be completed to a perfect covering. The red (color online) regions cannot be entirely covered by crosses.

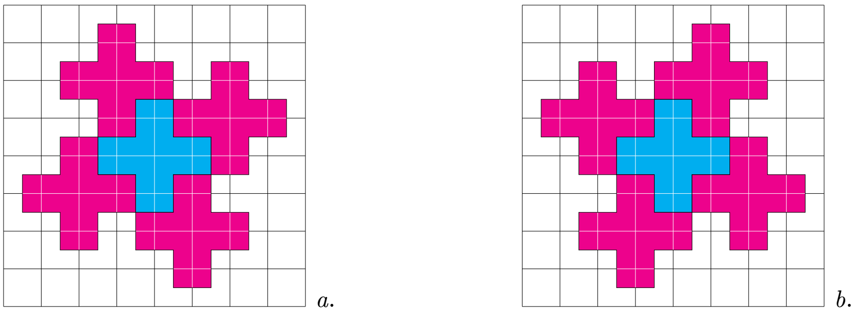

fig 2.5: The two configurations in $\mathbb{S}(\{x\}) \equiv \{X_a, X_b\}$. The cross at x is drawn in cyan (color online), whereas the crosses in $X_i \setminus \{x\}$ are drawn in magenta (color online). For each $i \in \{a, b\}$, there exists a unique μ_i such that $X_i \subset \mathcal{L}_{\mu_i}$.

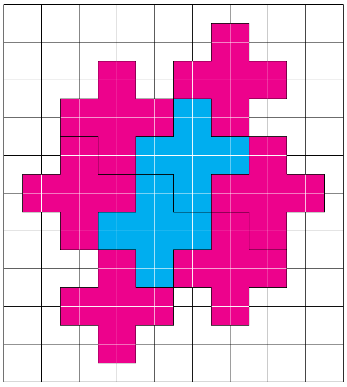

fig 2.6: If X is a pair of left-stacked crosses (in cyan, color online), then this is the unique configuration $X' \in \mathbb{S}(X)$. The crosses in $X' \setminus X$ are drawn in magenta (color online).

3 - By proceeding in a similar way, one proves that the models depicted in figure 2.7 are all non-sliding hard-core lattice particle systems. There are many more examples, among which the hard hexagon model (see figure 1.1c), and many more polyominoes than those depicted in figure 2.7. In addition, for every hard polyomino model (a cross is a polyomino) that is non-sliding, the corresponding model with a finer lattice mesh is also non-sliding.

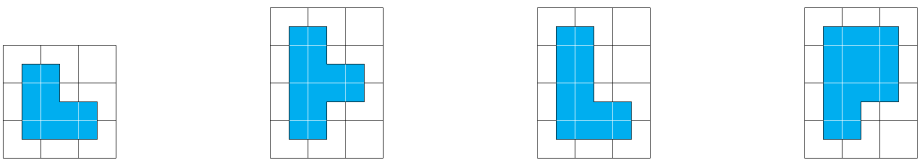

fig 2.7: More examples of non-sliding hard-core lattice particle systems. These shapes are all *polyominoes*.

3. High-fugacity expansion

In this section, we will prove the convergence of the high-fugacity expansion for non-sliding hard-core lattice particle systems. To that end, we will map the particle system to a model of *Gaunt-Fisher configurations* (GFc), and use a cluster expansion to compute the GFc partition function.

3.1. The GFc model

In this subsection, we map the particle system to a GFc model. The precise definition of the set of GFcs is given in definition 3.1. It is a bit technical, and the readers that are interested in the proof of lemma 3.2 are invited to skip definition 3.1 and come back to it as it appears in the lemma. An example is provided in figure 3.1.

Definition 3.1
(*Gaunt-Fisher configurations*)

Given a connected subset $\Gamma \subset \Lambda$, we denote the *exterior* of Γ by Γ_0, and its *holes* by $\mathcal{H}(\Gamma) \equiv \{\hat\Gamma_1, \cdots, \hat\Gamma_{h_\Gamma}\}$ with $h_\Gamma \geqslant 0$. Formally, $\hat\Gamma_0, \cdots, \hat\Gamma_{h_\Gamma}$ are the connected components of $\Lambda_\infty \setminus \Gamma$, and $\hat\Gamma_0$ is the only unbounded one.

Given $\nu \in \{1, \cdots, \tau\}$, a GFc is a quadruplet $\gamma \equiv (\Gamma_\gamma, X_\gamma, \nu, \underline{\mu}_\gamma)$ in which Γ_γ is a *connected* and *bounded* subset of Λ, $X_\gamma \in \Omega(\Gamma_\gamma)$, and $\underline{\mu}_\gamma$ is a map $\mathcal{H}(\Gamma_\gamma) \to \{1, \cdots, \tau\}$, and satisfies the following condition. Let $\mathfrak{X}_\gamma$ denote the particle configuration obtained by covering the exterior and holes of Γ_γ by particles:

$$\mathfrak{X}_\gamma := \left(\mathcal{L}_\nu \cap \hat\Gamma_{\gamma,0}\right) \cup \left(\bigcup_{j=1}^{h_{\Gamma_\gamma}} \left(\mathcal{L}_{\underline{\mu}_\gamma(\hat\Gamma_{\gamma,j})} \cap \hat\Gamma_{\gamma,j}\right)\right). \tag{3.1}$$

A quadruplet γ is a GFc if

- The particles in X_γ are entirely contained inside Γ_γ and those in $\mathfrak{X}_\gamma$ do not intersect Γ_γ: $\forall x \in X_\gamma$, $\sigma_x \subset \Gamma_\gamma$ and $\forall x' \in \mathfrak{X}_\gamma$, $\sigma_x \cap \Gamma_\gamma = \emptyset$.
- for every $x \in X_\gamma$, there exists $y \in \sigma_x$ (recall (1.3)) and $y' \in \mathcal{E}_\Lambda(X_\gamma \cup \mathfrak{X}_\gamma)$ (recall (1.4)) such that y and y' are neighbors,
- for every $x \in \mathfrak{X}_\gamma$ and $y \in \sigma_x$ and every $y' \in \mathcal{E}_\Lambda(X_\gamma \cup \mathfrak{X}_\gamma)$, y and y' are *not* neighbors.

We denote the set of GFcs by $\mathfrak{C}_\nu(\Lambda)$.

Lemma 3.2
(*GFc mapping*)

The partition function (1.27) can be rewritten as

$$\frac{\Xi_\Lambda^{(\nu)}(\underline{z})}{\mathbf{z}_\nu(\Lambda)} = \sum_{\underline{\gamma} \subset \mathfrak{C}_\nu(\Lambda)} \left(\prod_{\gamma \neq \gamma' \in \underline{\gamma}} \Phi(\gamma, \gamma')\right) \prod_{\gamma \in \underline{\gamma}} \zeta_\nu^{(\underline{z})}(\gamma) \tag{3.2}$$

where

$$\mathbf{z}_\nu(\Lambda) := \prod_{x \in \Lambda \cap \mathcal{L}_\nu} z(x) \tag{3.3}$$

and

$$\zeta_\nu^{(\underline{z})}(\gamma) := \frac{\prod_{x \in X_\gamma} z(x)}{\mathbf{z}_\nu(\Gamma_\gamma)} \prod_{j=1}^{h_{\Gamma_\gamma}} \frac{\Xi_{\hat\Gamma_{\gamma,j}}^{(\underline{\mu}_\gamma(\hat\Gamma_{\gamma,j}))}(\underline{z})}{\Xi_{\hat\Gamma_{\gamma,j}}^{(\nu)}(\underline{z})} \tag{3.4}$$

and $\Phi(\gamma,\gamma')\in\{0,1\}$ is equal to 1 if and only if Γ_γ and $\Gamma_{\gamma'}$ are disjoint.

Proof: We will first map particle configurations to a set of GFc, then extract the most external ones, and conclude the proof by induction.

1 - GFcs. To a configuration $X\in\Omega_\nu(\Lambda)$, we associate a set of *external GFcs*, in the following way, which is reminiscent of how Peierls' contours are associated to Ising spin configurations. See figure 3.1 for an example.

Given $x\in\Lambda$, let $\partial_X(x)$ denote the set of sites covered by particles neighboring x:

$$\partial_X(x):=\{y\in\Lambda_\infty,\quad \exists y'\in X,\ y\in\sigma_{y'},\ \sigma_{y'}\text{ neighbors }x\}. \tag{3.5}$$

Given $x\neq x'\in\mathcal{E}_\Lambda(X)$ (recall (1.4)), x and x' are said to be X*-neighbors* if they are either neighbors, or if $\partial_X(x)\cap\partial_X(x')\neq\emptyset$. This defines a natural notion of an X-connected subset of $\mathcal{E}_\Lambda(X)$: $E\subset\mathcal{E}_\Lambda(X)$ is X-connected if there exists a path $x_0,\cdots,x_n\in E$ such that $x_0\equiv x$, $x_n\equiv x'$ and for $i\in\{1,\cdots,n\}$, x_{i-1} and x_i are X-neighbors. Now, let $\{E_1,\cdots,E_n\}$ denote the set of X-connected components of $\mathcal{E}_\Lambda(X)$. We associate a set Γ_i to each E_i:

$$\Gamma_i:=E_i\cup\left(\bigcup_{x\in E_i}\partial_X(x)\right). \tag{3.6}$$

By construction, distinct Γ_i's are disjoint.

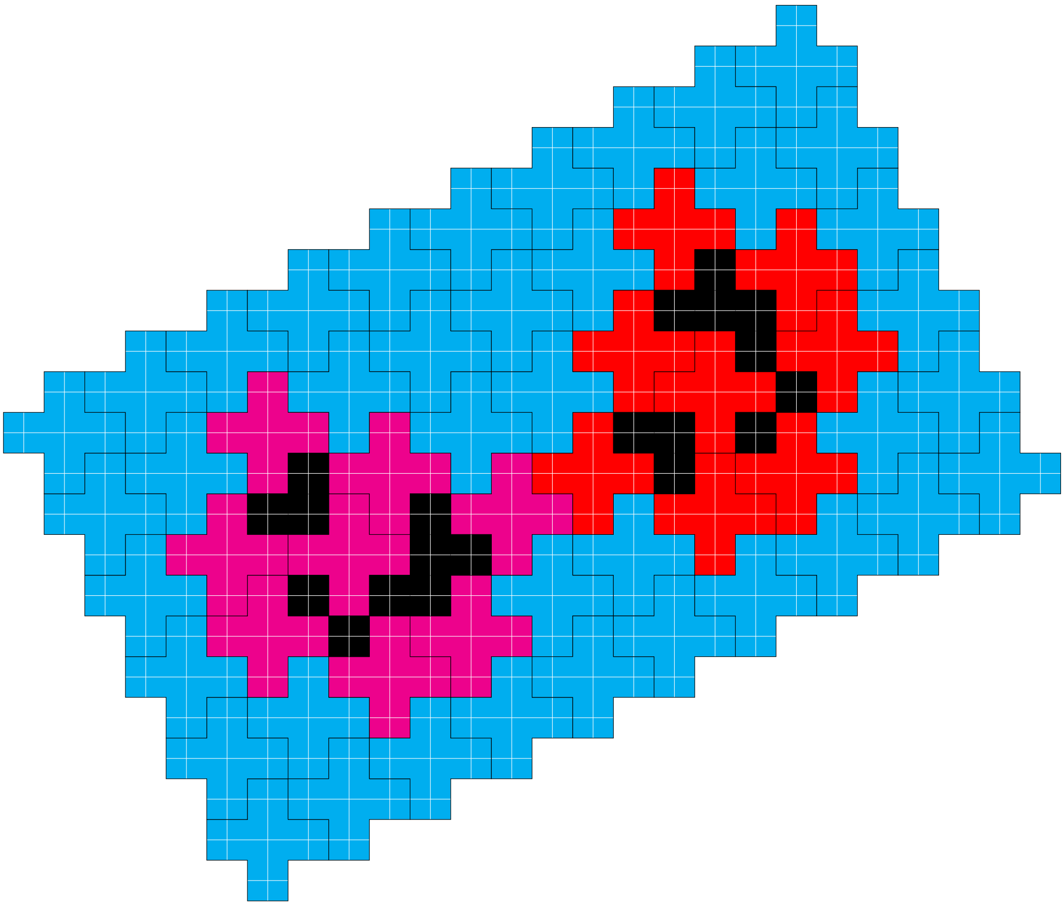

fig 3.1: An example cross configuration, and its associated GFc supports. There are two disconnected GFcs: the first consists of the red (color online) crosses and the neighboring black empty sites, and the second consists of the magenta (color online) crosses and the neighboring black empty sites.

We then denote the connected components of $\Lambda_\infty \setminus (\Gamma_1 \cup \cdots \cup \Gamma_n)$ by $\{\kappa_1, \cdots, \kappa_m\}$. By construction, each κ_i is covered by particles, which we denote by X_i, and

$$\bar{X}_i := X_i \cup \{x \in X,\ \exists x' \in X_i,\ \Delta(\sigma_x, \sigma_{x'}) = 1\} \in \mathbb{S}(X_i) \tag{3.7}$$

(we recall that Δ is the graph distance on Λ_∞, and that $\mathbb{S}$ was defined in definition 1.1) so that, by the non-sliding condition, there exists a *unique* $\mu_i \in \{1, \cdots, \tau\}$ such that $\bar{X}_i \subset \mathcal{L}_{\mu_i}$. See figure 3.2 for an example.

By construction, for every $i \in \{1, \cdots, n\}$, the holes of Γ_i, which, we recall, are denoted by $\hat{\Gamma}_{i,j}$, contain at least one of the κ_k. When a hole contains several κ_k's, there is one that is *more external* than the others (see figure 3.2). Let us now make this idea more precise, and use it to define the GFc configuration associated to X. For every $i \in \{1, \cdots, n\}$ and $j \in \{0, \cdots, h_{\Gamma_i}\}$, we define $k(\hat{\Gamma}_{i,j}) \in \{1, \cdots, m\}$ such that, for every $i' \in \{1, \cdots, n\}$ and $j' \in \{1, \cdots, h_{\Gamma_{i'}}\}$, $\kappa_{k(\hat{\Gamma}_{i,j})} \cap \hat{\Gamma}_{i',j'} \neq \emptyset$ if and only if $(i, j) = (i', j')$. We then define the set of GFcs associated to X as the set of quadruplets

$$\underline{\gamma}(X) = \left\{ \left(\Gamma_i, X \cap \Gamma_i,\ \mu_{k(\hat{\Gamma}_{i,0})},\ \underline{\mu}_i \right), \quad i \in \{1, \cdots, n\} \right\} \tag{3.8}$$

where $X \cap \Gamma_i$ is the restriction of the particle configuration to Γ_i, and $\underline{\mu}_i$ is the map from $\mathcal{H}(\hat{\Gamma}_i)$ to $\{1, \cdots, \tau\}$ defined by

$$\underline{\mu}_i(\hat{\Gamma}_{i,j}) = \mu_{k(\hat{\Gamma}_{i,h_{\Gamma_i}})}. \tag{3.9}$$

The set of quadruplets thus constructed is a set of GFcs, in the sense of definition 3.1.

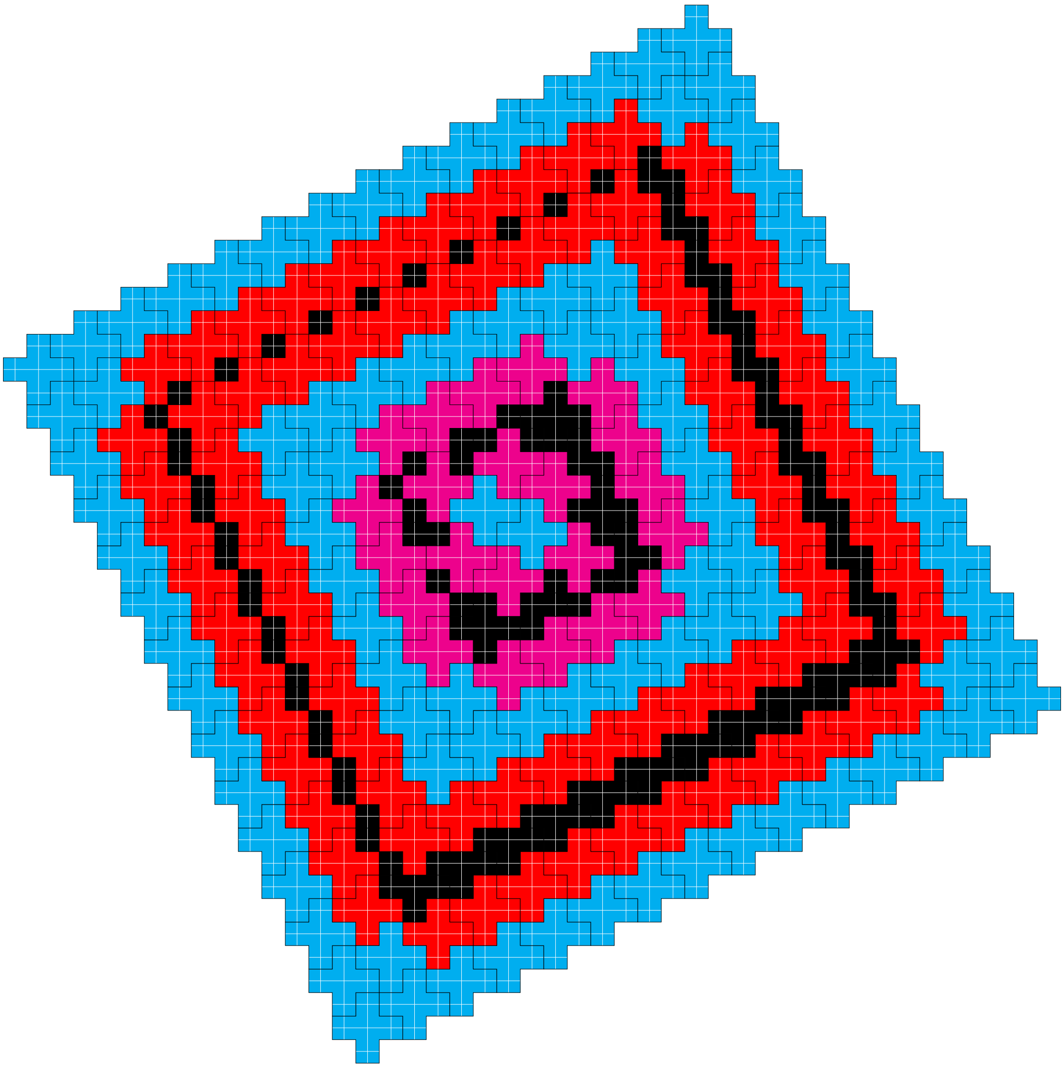

fig 3.2: A configuration in which the GFc supports are nested. The κ_i are the connected components of cyan (color online) crosses. Each is a subset of a unique perfect covering.

2 - External GFc model. We have thus mapped X to a model of GFcs. Note that the indices $\mu_\cdot$ must match up, that is, if a GFc is in the hole of another, its external μ must be equal to the μ of the hole it is in. This is a long range interaction between GFcs, which makes the GFc model difficult to study. Instead, we will map the system to a model of *external* GFcs, that do not have long range interactions. We introduce the following definitions: two GFcs $\gamma, \gamma' \in \mathfrak{C}_\nu(\Lambda)$ are said to be

- *compatible* if their supports do not intersect, that is, $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$,
- *external* if their supports are in each other's exteriors, that is, $\Gamma_\gamma \subset \hat{\Gamma}_{\gamma',0}$ and $\Gamma_{\gamma'} \subset \hat{\Gamma}_{\gamma,0}$.

The GFcs in $\underline{\gamma}(X)$ (see (3.8)) are compatible, but not necessarily external. Roughly, the idea is to keep the GFcs that are external, since those do not have long-range interactions (they all share the same external μ, which is fixed to ν once and for all). At that point, the particle configuration in the exterior of all GFcs is fixed, and we are left with summing over configurations in the holes. The sum over configurations in holes is of the same form as (1.27), with Λ replaced by the hole, and the boundary condition by the appropriate $\underline{\mu}$. Following this, we rewrite (1.27) as

$$\frac{\Xi_\Lambda^{(\nu)}(\underline{z})}{\mathbf{z}_\nu(\Lambda)} = \sum_{\underline{\gamma} \subset \mathfrak{C}_\nu(\Lambda)} \left(\prod_{\gamma \neq \gamma' \in \underline{\gamma}} \Phi_{\mathrm{ext}}(\gamma, \gamma') \right) \prod_{\gamma \in \underline{\gamma}} \left(\frac{\prod_{x \in X_\gamma} z(x)}{\mathbf{z}_\nu(\Gamma_\gamma)} \prod_{j=1}^{h_{\Gamma_\gamma}} \frac{\Xi_{\hat{\Gamma}_{\gamma,j}}^{(\underline{\mu}_\gamma(\hat{\Gamma}_{\gamma,j}))}(\underline{z})}{\mathbf{z}_\nu(\hat{\Gamma}_{\gamma,j})} \right) \tag{3.10}$$

in which $\Phi_{\mathrm{ext}}(\gamma,\gamma')\in\{0,1\}$ is equal to 1 if and only if γ and γ' are *compatible* and *external*. We have, thus, rewritten the model as a system of external GFcs.

3 - GFc model. The last factor in (3.10) is similar to the left side of (3.10), except for the fact that the boundary condition is $\underline{\mu}_\gamma(\hat\Gamma_{\gamma,j})$ instead of ν. In order to obtain a model of GFcs (which are not necessarily external), we could iterate (3.10), but, as was discussed earlier, this would induce long-range correlations. Instead, we introduce a trivial identity into (3.10):

$$\frac{\Xi_\Lambda^{(\nu)}(\underline{z})}{\mathbf{z}_\nu(\Lambda)}=\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\left(\prod_{\gamma\neq\gamma'\in\underline{\gamma}}\Phi_{\mathrm{ext}}(\gamma,\gamma')\right)\prod_{\gamma\in\underline{\gamma}}\left(\zeta_\nu^{(\underline{z})}(\gamma)\prod_{j=1}^{h_{\Gamma_\gamma}}\frac{\Xi_{\hat\Gamma_{\gamma,j}}^{(\nu)}(\underline{z})}{\mathbf{z}_\nu(\hat\Gamma_{\gamma,j})}\right)\tag{3.11}$$

in which $\zeta_\nu^{(\underline{z})}(\gamma)$ is defined in (3.4). We then rewrite $\Xi_{\hat\Gamma_{\gamma,j}}^{(\nu)}(\underline{z})$ using (3.11), iterate, and, noting that, if $\Gamma_{\gamma,j}$ does not contain GFcs, then $\Xi_{\hat\Gamma_{\gamma,j}}^{(\nu)}(z)=\mathbf{z}_\nu(\hat\Gamma_{\gamma,j})$, we find (3.2). □

3.2. Cluster expansion of the GFc model

As was discussed in section 1.2, the pressure of a system of hard particles at low fugacity can be expressed as a convergent power series. The GFc model in (3.2) is a system of hard GFcs (the factor $\Phi(\gamma,\gamma')$ is a hard-core interaction), and, as we will see below, the GFcs have a small activity. Similarly to the low-fugacity expansion, the logarithm of the left side of (3.2) can be expressed as a convergent power series. In this context, in which the hard GFcs have more structure than hard particles, the expansion is usually called a *cluster expansion*. The cluster expansion has been studied extensively (to cite but a few [Ru99, GBG04, KP86, BZ00]), and we will use a theorem by Bovier and Zahradnik [BZ00], which is summarized in the following lemma.

Lemma 3.3

(*convergence of the cluster expansion* [BZ00])

If there exist two functions a,d that map $\mathfrak{C}_\nu(\Lambda)$ to $[0,\infty)$ and a number $\delta\geqslant 0$, such that $\forall\gamma\in\mathfrak{C}_\nu(\Lambda)$,

$$|\zeta_\nu^{(\underline{z})}(\gamma)|e^{a(\gamma)+d(\gamma)}\leqslant\delta<1,\qquad\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda)\\ \gamma'\not\sim\gamma}}|\zeta_\nu^{(\underline{z})}(\gamma')|e^{a(\gamma')+d(\gamma')}\leqslant\frac{\delta}{|\log(1-\delta)|}a(\gamma)\tag{3.12}$$

in which $\gamma'\not\sim\gamma$ means that γ' and γ are *not* compatible (that is, their supports overlap), then

$$\frac{\Xi_\Lambda^{(\nu)}(\Lambda)}{\mathbf{z}_\nu(\Lambda)}=\exp\left(\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma)\right)\tag{3.13}$$

in which Φ^T is the *Ursell function*, defined as

$$\Phi^T(\gamma_1,\cdots,\gamma_n):=\sum_{\mathfrak{g}\in\mathcal{G}^T(n)}\prod_{\{j,j'\}\in\mathcal{E}(\mathfrak{g})}(\Phi(\gamma_j,\gamma_{j'})-1)\tag{3.14}$$

where $\Phi(\gamma_j,\gamma_{j'})\in\{0,1\}$ is equal to 1 if and only if Γ_{γ_j} and $\Gamma_{\gamma_{j'}}$ are *disjoint*, $\mathcal{G}^T(n)$ is the set of connected graphs on n vertices and $\mathcal{E}(\mathfrak{g})$ is the set of edges of $\mathfrak{g}$. In addition, for every $\gamma\in\mathfrak{C}_\nu(\Lambda)$,

$$\sum_{\underline{\gamma}'\subset\mathfrak{C}_\nu(\Lambda)}\left|\Phi^T(\gamma\cup\underline{\gamma}')\prod_{\gamma'\in\underline{\gamma}'}\left(\zeta_\nu^{(\underline{z})}(\gamma')e^{d(\gamma')}\right)\right|\leqslant e^{a(\gamma)}.\tag{3.15}$$

We will now show that (3.12) holds for an appropriate choice of a, d and δ.

Lemma 3.4

(*bound on the activity*)

Let

$$\mathcal{N} := \sup_{x\in\Lambda_\infty, X\in\Omega(\Lambda_\infty)} |\partial_X(x)|. \tag{3.16}$$

If $z(x) \equiv z$ for every $x \in \Lambda_\infty$ except for a finite number $\mathfrak{n}$ of sites $(\mathfrak{x}_1, \cdots, \mathfrak{x}_\mathfrak{n})$, and if there exist $z_0, c_1 > 0$ such that $|z| > z_0$ and

$$e^{-\frac{c_1}{\mathfrak{n}}}|z| \leqslant |z(\mathfrak{x}_i)| \leqslant e^{\frac{c_1}{\mathfrak{n}}}|z| \tag{3.17}$$

then, for every $\theta, \xi \in (0,1)$ such that $\theta + \xi < 1$, (3.12) is satisfied with

$$a(\gamma) := -\theta|\Gamma_\gamma| \log\alpha > 0, \quad d(\gamma) := -\xi|\Gamma_\gamma| \log\alpha > 0 \tag{3.18}$$

in which

$$\alpha := e^1|z|^{-\rho_m(1+\mathcal{N})^{-1}} \ll 1. \tag{3.19}$$

In addition, there exists $C_1 \in (0,\xi)$ such that, for every $i \in \{1, \cdots, \mathfrak{n}\}$, and every $\mu \in \{1, \cdots, \tau\}$

$$\left|\frac{\partial}{\partial \log z(\mathfrak{x}_i)} \log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)\right| \leqslant \alpha^{C_1}\mathbb{1}(\mathfrak{x}_i \in \Lambda) \tag{3.20}$$

in which $\mathbb{1}(E) \in \{0,1\}$ is equal to 1 if and only if E is true.

Remark: The value of z_0 depends on the model. It is worked out rather explicitly in the proof, and appears as a smallness condition on α, which is made explicit in (3.31), (3.34), (3.36), and -(3.47). In these equations, we use the notation $\alpha \ll (\cdots)$ to mean "there exists a constant $c > 0$ such that if $\alpha < c(\cdots)$".

Proof: We will prove this lemma along with the following inequality: $\exists\varsigma > 0$ such that, for every $\mu \in \{1, \cdots, \tau\}$

$$\left|\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\Xi_\Lambda^{(\nu)}(\underline{z})}\right| \leqslant \varsigma e^{|\partial\Lambda|} \tag{3.21}$$

in which $\partial\Lambda$ is the set of sites in Λ that neighbor $\Lambda_\infty \setminus \Lambda$. We proceed by induction on the volume $|\Lambda|$ of Λ. (Note that, for certain models, this ratio is identically equal to 1. This is the case when the different perfect coverings are related to each other by a translation, as in the hard diamond model. However, for the hard-cross model, in which certain perfect coverings are related by a reflection, the ratio may differ from 1, see figure 3.3.)

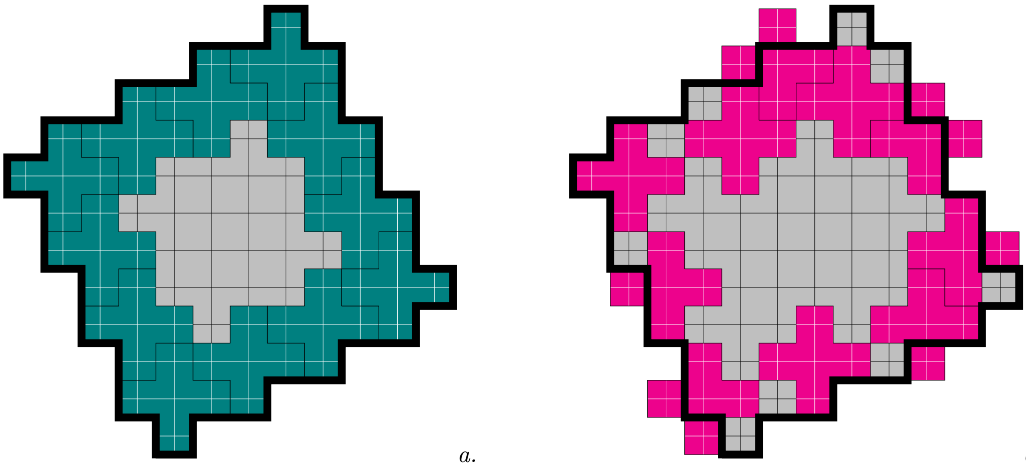

fig 3.3: Two different boundary conditions for the hard-cross model. The set Λ is outlined by the thick black line. The crosses that are drawn are those mandated by the boundary condition (the boundary condition stipulates that every cross that is in contact with the boundary must be of a specified phase), and the remaining available space in Λ is colored gray. The partition function in the case of figure *a* is

$$z^{16}(1+4y+10y^2+8y^3+y^4)$$

whereas that in figure *b* is

$$z^{16}(1+6y+18y^2+48y^3+43y^4+13y^5+y^6).$$

1 - First of all, if Λ is so small that it cannot contain a GFc, that is, $\mathcal{C}_\mu(\Lambda)=\emptyset$ for every $\mu\in\{1,\cdots,\tau\}$, then (3.12) is trivially true, and

$$\Xi_\Lambda^{(\mu)}(\underline{z})=\mathbf{z}_\mu(\Lambda)=\prod_{x\in\Lambda\cap\mathcal{L}_\mu}z(x). \tag{3.22}$$

Therefore, (3.20) holds. We now turn to (3.21). The x dependence of $z(x)$ can be neglected, since there can be at most $\mathfrak{n}$ factors that differ from z, and they do so by a bounded amount:

$$e^{-c_1}|z|^{|\Lambda\cap\mathcal{L}_\mu|}\leqslant|\Xi_\Lambda^{(\mu)}(\underline{z})|\leqslant e^{c_1}|z|^{|\Lambda\cap\mathcal{L}_\mu|}. \tag{3.23}$$

In addition, as we will show below, $|\Lambda\cap\mathcal{L}_\mu|$ is independent of μ, which implies that

$$\left|\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\Xi_\Lambda^{(\nu)}(\underline{z})}\right|\leqslant e^{2c_1}\leqslant\varsigma e^{|\partial\Lambda|} \tag{3.24}$$

provided

$$\varsigma\geqslant e^{2c_1}. \tag{3.25}$$

So, to conclude this argument, it suffices to prove that $|\Lambda\cap\mathcal{L}_\mu|$ is independent of μ. This follows from the fact that Λ is *tiled* (see (1.25)). In fact, we will show that for every $x\in\Lambda_\infty$, $|\mathcal{L}_\mu\cap\sigma_x|=1$ for any μ, which, by (1.25) implies that $|\Lambda\cap\mathcal{L}_\mu|=\rho_m|\Lambda|$. We proceed in two steps, by first showing that $|\mathcal{L}_\mu\cap\sigma_x|$ is smaller than 2, and then that it is larger than 0.

- To prove that $|\mathcal{L}_\mu\cap\sigma_x|<2$, we show that if $y,y'\in\mathcal{L}_\mu\cap\sigma_x$, then $\sigma_y\cap\sigma_{y'}\neq\emptyset$. Indeed, since $y\in\sigma_x$, writing $y'=x+v\in\sigma_x$, by translating by v, we find that $\sigma_{y'}\equiv\sigma_{x+v}\ni y+v\in\sigma_y$. Therefore, $|\mathcal{L}_\mu\cap\sigma_x|\leqslant 2$.
- Finally, if $|\mathcal{L}_\mu\cap\sigma_x|=0$, then, since $\mathcal{L}_\mu$ is periodic, the density of $\mathcal{L}_\mu$ would be $<\rho_m$, which contradicts the fact that the $\mathcal{L}_i$ are related to each other by isometries.

All in all, $|\mathcal{L}_\mu \cap \sigma_x| = 1$,which concludes the proof of (3.24).

2 - From now on, we assume that (3.21) holds for every tiled strict subset of Λ (note that $\hat\Gamma_{\gamma,j}$ is a tiled strict subset of Λ). We first prove (3.12).

2-1 - By (3.4) and (3.21),

$$|\zeta_\nu^{(z)}(\gamma)| \leqslant e^{2c_1}\varsigma \frac{|z|^{|X_\gamma|}}{|z|^{\rho_m|\Gamma_\gamma|}} e^{|\Gamma_\gamma|}. \tag{3.26}$$

By definition 3.1, in every configuration X_γ, every particle must be in contact with at least one empty site. Therefore, the fraction $\psi_\gamma(X_\gamma)$ of empty sites in Γ_γ must satisfy

$$\psi_\gamma(X_\gamma) := \frac{|\mathcal{E}_{\Gamma_\gamma}(X_\gamma)|}{|\Gamma_\gamma|} \geqslant \frac{1}{\mathcal{N}+1} \tag{3.27}$$

(recall that $\mathcal{E}_{\Gamma_\gamma}(X_\gamma)$ is the number of empty sites (1.4), and $\mathcal{N}$ is the maximal volume occupied by particles that neighbor a site (3.16)). Therefore,

$$|X_\gamma| = \rho_m|\Gamma_\gamma|(1-\psi_\gamma(X_\gamma)) \leqslant \rho_m|\Gamma_\gamma|\frac{\mathcal{N}}{\mathcal{N}+1}. \tag{3.28}$$

Therefore,

$$|\zeta_\nu^{(z)}(\gamma)| \leqslant \varsigma^2 \left(e^1|z|^{-\rho_m\frac{1}{\mathcal{N}+1}}\right)^{|\Gamma_\gamma|} \equiv \varsigma^2\alpha^{|\Gamma_\gamma|}. \tag{3.29}$$

Thus,

$$|\zeta_\nu^{(z)}(\gamma)|e^{a(\gamma)+d(\gamma)} \leqslant \varsigma^2\alpha^{(1-(\theta+\xi))|\Gamma_\gamma|} \tag{3.30}$$

which proves the first of (3.12) with $\delta \equiv \varsigma^2\alpha^{1-(\theta+\xi)}$, which, provided

$$\alpha \ll \varsigma^{-2(1-(\theta+\xi))^{-1}} \tag{3.31}$$

is small.

2-2 - By (3.30),

$$\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda)\\ \gamma'\not\sim\gamma}} e^{a(\gamma')+d(\gamma')}|\zeta_\nu^{(z)}(\gamma')| \leqslant \varsigma^2 \sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda)\\ \gamma'\not\sim\gamma}} \alpha^{(1-(\theta+\xi))|\Gamma_{\gamma'}|}. \tag{3.32}$$

We bound the number of GFcs γ' that are *incompatible* with a fixed GFc γ by the number of walks on Λ_∞ of length $2|\Gamma_{\gamma'}| \equiv 2\ell$ that intersect Γ_γ:

$$\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda)\\ \gamma'\not\sim\gamma}} e^{a(\gamma')+d(\gamma')}|\zeta_\nu^{(z)}(\gamma')| \leqslant \varsigma^2|\Gamma_\gamma|\sum_{\ell=1}^\infty \chi^{2\ell}\alpha^{(1-(\theta+\xi))\ell} \tag{3.33}$$

in which χ is the degree of Λ_∞, and, provided

$$\alpha \ll \chi^{-2(1-(\theta+\xi))^{-1}} \tag{3.34}$$

we have

$$\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda)\\ \gamma'\not\sim\gamma}} e^{a(\gamma')+d(\gamma')}|\zeta_\nu^{(z)}(\gamma')| \leqslant \varsigma^2 c_2|\Gamma_{\gamma'}| \tag{3.35}$$

for some constant $c_2 > 0$. If, in addition,

$$\alpha \ll e^{-\varsigma^2 c_2\theta^{-1}} \tag{3.36}$$

then this implies (3.12).

3 - Let us now prove (3.20). Since (3.12) holds, the cluster expansion in lemma 3.3 is absolutely convergent. Thus, by (3.13),

$$\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)=\sum_{\gamma'\in\mathfrak{C}_\mu(\Lambda)}\frac{\partial\zeta_\mu^{(\underline{z})}(\gamma')}{\partial\log z(\mathfrak{x}_i)}\sum_{\underline{\gamma}\subset\mathfrak{C}_\mu(\Lambda)}\Phi^T(\gamma'\cup\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\mu^{(\underline{z})}(\gamma)\tag{3.37}$$

so, by (3.15),

$$\left|\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)\right|\leqslant\sum_{\gamma'\in\mathfrak{C}_\mu(\Lambda)}e^{a(\gamma')}\left|\frac{\partial\zeta_\mu^{(\underline{z})}(\gamma')}{\partial\log z(\mathfrak{x}_i)}\right|.\tag{3.38}$$

Furthermore, by (3.4),

$$\begin{aligned}\frac{\partial\log\zeta_\mu^{(\underline{z})}(\gamma')}{\partial\log z(\mathfrak{x}_i)}&=\mathbb{1}\left(\mathfrak{x}_i\in X_{\gamma'}\right)-\mathbb{1}\left(\mathfrak{x}_i\in\mathcal{L}_\mu\cap\Gamma_{\gamma'}\right)\\&+\sum_{j=1}^{h_{\Gamma_{\gamma'}}}\left(\mathbb{1}\left(\mathfrak{x}_i\in\mathcal{L}_{\underline{\mu}_{\gamma'}(\hat\Gamma_{\gamma',j})}\cap\hat\Gamma_{\gamma',j}\right)-\mathbb{1}\left(\mathfrak{x}_i\in\mathcal{L}_\mu\cap\hat\Gamma_{\gamma',j}\right)\right)\\&+\sum_{j=1}^{h_{\Gamma_{\gamma'}}}\left(\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_{\hat\Gamma_{\gamma',j}}^{(\underline{\mu}_{\gamma'}(\hat\Gamma_{\gamma',j}))}(\underline{z})}{\mathbf{z}_{\underline{\mu}_{\gamma'}(\hat\Gamma_{\gamma',j})}(\hat\Gamma_{\gamma',j})}\right)-\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_{\hat\Gamma_{\gamma',j}}^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\hat\Gamma_{\gamma',j})}\right)\right).\end{aligned}\tag{3.39}$$

Therefore, using (3.20) inductively to estimate the last term,

$$\left|\frac{\partial\zeta_\mu^{(\underline{z})}(\gamma')}{\partial\log z(\mathfrak{x}_i)}\right|\leqslant|\zeta_\mu^{(\underline{z})}(\gamma')|3\mathbb{1}(\mathfrak{x}_i\in\mathrm{Int}(\Gamma_{\gamma'}))\tag{3.40}$$

in which

$$\mathrm{Int}(\Gamma_{\gamma'}):=\Gamma_{\gamma'}\cup\left(\bigcup_{j=1}^{h_{\Gamma_{\gamma'}}}\hat\Gamma_{\gamma',j}\right)\tag{3.41}$$

so that

$$\left|\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)\right|\leqslant3\sum_{\substack{\gamma'\in\mathfrak{C}_\mu(\Lambda)\\\mathrm{Int}(\Gamma_{\gamma'})\ni\mathfrak{x}_i}}e^{a(\gamma')}|\zeta_\mu^{(\underline{z})}(\gamma')|.\tag{3.42}$$

In addition, by the isoperimetric inequality,

$$|\mathrm{Int}(\Gamma_{\gamma'})|\leqslant c_3^{(d)}|\Gamma_{\gamma'}|^d\tag{3.43}$$

for some constant $c_3^{(d)}>0$ (which depends on d), so

$$\left|\frac{\partial}{\partial\log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)\right|\leqslant3\sum_{\substack{\gamma'\in\mathfrak{C}_\mu(\Lambda)\\\Gamma_{\gamma'}\ni\mathfrak{x}_i}}c_3^{(d)}|\Gamma_{\gamma'}|^de^{a(\gamma')}|\zeta_\mu^{(\underline{z})}(\gamma')|.\tag{3.44}$$

Furthermore,

$$|\Gamma_{\gamma'}|^d\leqslant d!e^{|\Gamma_{\gamma'}|}\tag{3.45}$$

so, rewriting

$$e^{a(\gamma')+|\Gamma_{\gamma'}|}=e^{-\bar d(\gamma')}e^{(a(\gamma')+d(\gamma'))},\quad\bar d(\gamma'):=d(\gamma)-|\Gamma_{\gamma'}|\geqslant-\xi\log\alpha-1\tag{3.46}$$

which holds provided

$$\alpha \leqslant e^{-\frac{1}{\xi}} \tag{3.47}$$

and by (3.35), we find

$$\left|\frac{\partial}{\partial \log z(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\mathbf{z}_\mu(\Lambda)}\right)\right| \leqslant \alpha^\xi 3e^1 c_3^{(d)} d!\varsigma^2 c_2. \tag{3.48}$$

which, provided

$$\alpha \leqslant \left(3e^1 c_3^{(d)} d!\varsigma^2 c_2\right)^{-(\xi - C_1)^{-1}} \tag{3.49}$$

implies (3.20).

4 - We now turn to the proof of (3.21).

4-1 - First of all, we get rid of the dependence on $z(\mathfrak{x}_i)$: by Taylor's theorem,

$$\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{z})}{\Xi_\Lambda^{(\nu)}(\underline{z})}\right) = \log\left(\frac{\Xi_\Lambda^{(\mu)}(z)}{\Xi_\Lambda^{(\nu)}(z)}\right) + \sum_{i=1}^{\mathfrak{n}}(\underline{z}(\mathfrak{x}_i) - z)\frac{\partial}{\partial \underline{\tilde{z}}(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{\tilde{z}})}{\Xi_\Lambda^{(\nu)}(\underline{\tilde{z}})}\right) \tag{3.50}$$

in which $\underline{\tilde{z}}$ is a function satisfying $\underline{\tilde{z}}(\mathfrak{x}_i) \in [z, \underline{z}(\mathfrak{x}_i)]$ and $\underline{\tilde{z}}(x) = z$ for any $x \neq \mathfrak{x}_i$. By (3.20),

$$\left|\frac{\partial}{\partial \underline{\tilde{z}}(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{\tilde{z}})}{\Xi_\Lambda^{(\nu)}(\underline{\tilde{z}})}\right)\right| \leqslant \frac{1}{|\underline{\tilde{z}}(\mathfrak{x}_i)|}\left(|\mathbb{1}\left(\mathfrak{x}_i \in \mathcal{L}_\mu \cap \Lambda\right) - \mathbb{1}\left(\mathfrak{x}_i \in \mathcal{L}_\nu \cap \Lambda\right)| + \alpha^{C_1}.\right) \tag{3.51}$$

Thus,

$$\left|\sum_{i=1}^{\mathfrak{n}}(\underline{z}(\mathfrak{x}_i) - z)\frac{\partial}{\partial \underline{\tilde{z}}(\mathfrak{x}_i)}\log\left(\frac{\Xi_\Lambda^{(\mu)}(\underline{\tilde{z}})}{\Xi_\Lambda^{(\nu)}(\underline{\tilde{z}})}\right)\right| \leqslant 2\mathfrak{n}(e^{\frac{2c_1}{\mathfrak{n}}} + 1). \tag{3.52}$$

4-2 - We now focus on $\Xi_\Lambda^{(\mu)}(z)$, and make use of the cluster expansion in lemma 3.3: by (3.13),

$$\log\left(\frac{\Xi_\Lambda^{(\mu)}(z)}{\Xi_\Lambda^{(\nu)}(z)}\right) = \sum_{\underline{\gamma} \subset \mathfrak{C}_\mu(\Lambda)} \Phi^T(\underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_\mu^{(z)}(\gamma) - \sum_{\underline{\gamma} \subset \mathfrak{C}_\nu(\Lambda)} \Phi^T(\underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_\nu^{(z)}(\gamma) \tag{3.53}$$

(we recall that $z^{|\Lambda \cap \mathcal{L}_\mu|}$ is independent of μ so the $\mathbf{z}_\mu(\Lambda)$ and $\mathbf{z}_\nu(\Lambda)$ factors cancel out). We then split these cluster expansions into *bulk* and *boundary* contributions, which are defined as follows. Let $\mathfrak{C}_\mu^{(|\Lambda|)}(\Lambda_\infty)$ denote the set of GFcs in Λ_∞ whose upper-leftmost corner (if $d > 2$, then this notion should be extended in the obvious way) is in Λ. Note that $\mathfrak{C}_\mu^{(|\Lambda|)}(\Lambda_\infty)$ only depends on Λ through its cardinality $|\Lambda|$. We then write

$$\sum_{\underline{\gamma} \subset \mathfrak{C}_\mu(\Lambda)} \Phi^T(\underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_\mu^{(z)}(\gamma) = \mathfrak{B}_\mu^{(|\Lambda|)}(\Lambda_\infty) - \mathfrak{b}_\mu^{(\Lambda)}(\Lambda_\infty) \tag{3.54}$$

in which $\mathfrak{B}$ is the *bulk* contribution, and $\mathfrak{b}$ is the *boundary* term.

$$\begin{aligned}
\mathfrak{B}_\mu^{(|\Lambda|)}(\Lambda_\infty) &:= \sum_{\gamma' \in \mathfrak{C}_\mu^{(|\Lambda|)}(\Lambda_\infty)} \zeta_\mu^{(z)}(\gamma') \sum_{\underline{\gamma} \subset \mathfrak{C}_\mu(\Lambda_\infty)} \Phi^T(\gamma' \cup \underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_\mu^{(z)}(\gamma) \\
\mathfrak{b}_\mu^{(\Lambda)}(\Lambda_\infty) &:= \sum_{\gamma' \in \mathfrak{C}_\mu^{(|\Lambda|)}(\Lambda_\infty)} \zeta_\mu^{(z)}(\gamma') \sum_{\substack{\underline{\gamma} \subset \mathfrak{C}_\mu(\Lambda_\infty) \\ \gamma' \cup \underline{\gamma} \not\subset \mathfrak{C}_\mu(\Lambda)}} \Phi^T(\gamma' \cup \underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_\mu^{(z)}(\gamma).
\end{aligned} \tag{3.55}$$

4-2-1 - The bulk terms cancel each other out. Indeed, we recall (see section 1.1) that there exists an isometry $F_{\mu,\nu}$ of Λ_∞ such that $F_{\mu,\nu}(\mathcal{L}_\mu)=\mathcal{L}_\nu$. In addition, since $F_{\mu,\nu}$ is an isometry, it maps perfect coverings to perfect coverings, and this map is denoted by $f_{\mu,\nu}:\{1,\cdots,\tau\}\to\{1,\cdots,\tau\}$:

$$\mathcal{L}_{f_{\mu,\nu}(\kappa)}=F_{\mu,\nu}(\mathcal{L}_\kappa). \tag{3.56}$$

This allows us to define an action on GFcs: $\mathfrak{F}_{\mu,\nu}:\mathfrak{C}_\mu(\Lambda)\to\mathfrak{C}_\nu(F_{\mu,\nu}(\Lambda))$,

$$\mathfrak{F}_{\mu,\nu}(\Gamma_\gamma,X_\gamma,\mu,\underline{\mu}_\gamma):=(F_{\mu,\nu}(\Gamma_\gamma),F_{\mu,\nu}(X_\gamma),\nu,f_{\mu,\nu}(\underline{\mu}_\gamma)). \tag{3.57}$$

The map $\mathfrak{F}_{\mu,\nu}$ is a bijection and, since the partition function is invariant under isometries, it leaves $\zeta_\mu^{(z)}$ and Φ^T invariant, so

$$\mathfrak{B}_\mu^{(|\Lambda|)}(\Lambda_\infty)=\sum_{\gamma'\in\mathfrak{C}_\nu^{(|F_{\mu,\nu}(\Lambda)|)}(F_{\mu,\nu}(\Lambda_\infty))}\zeta_\nu^{(z)}(\gamma')\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(F_{\mu,\nu}(\Lambda_\infty))}\Phi^T(\gamma'\cup\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(z)}(\gamma) \tag{3.58}$$

so, since $F_{\mu,\nu}(\Lambda_\infty)=\Lambda_\infty$ and $|F_{\mu,\nu}(\Lambda)|=|\Lambda|$,

$$\mathfrak{B}_\mu^{(|\Lambda|)}(\Lambda_\infty)-\mathfrak{B}_\nu^{(|\Lambda|)}(\Lambda_\infty)=0. \tag{3.59}$$

4-2-2 - Finally, we estimate the boundary term. First of all, since every cluster $\gamma'\cup\underline{\gamma}$ that is not a subset of $\mathfrak{C}_\mu(\Lambda)$ must contain at least one GFc that goes over the boundary of Λ,

$$\mathfrak{b}_\mu^{(\Lambda)}(\Lambda_\infty)\leqslant\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda_\infty)\\ \Gamma_{\gamma'}\cap\Lambda\neq\emptyset\\ \Gamma_{\gamma'}\cap(\Lambda_\infty\setminus\Lambda)\neq\emptyset}}|\zeta_\mu^{(z)}(\gamma')|\sum_{\underline{\gamma}\subset\mathfrak{C}_\mu(\Lambda_\infty)}\left|\Phi^T(\gamma'\cup\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\mu^{(z)}(\gamma)\right| \tag{3.60}$$

so, by (3.15),

$$|\mathfrak{b}_\mu^{(\Lambda)}(\Lambda_\infty)|\leqslant\sum_{\substack{\gamma\in\mathfrak{C}_\nu(\Lambda_\infty)\\ \Gamma_\gamma\cap\Lambda\neq\emptyset\\ \Gamma_\gamma\cap(\Lambda_\infty\setminus\Lambda)\neq\emptyset}}|\zeta_\mu^{(z)}(\gamma')|e^{a(\gamma')} \tag{3.61}$$

which, rewriting, as we did earlier $e^{a(\gamma')}=e^{-d(\gamma')}e^{a(\gamma')+d(\gamma')}$ and using $d(\gamma')\geqslant-\xi\log\alpha$, implies

$$|\mathfrak{b}_\mu^{(\Lambda)}(\Lambda_\infty)|\leqslant\alpha^\xi\varsigma^2c_2|\partial\Lambda|. \tag{3.62}$$

4-2-3 - Thus, inserting (3.59) and (3.62) into (3.54) and (3.53), we find that, provided - (3.36) holds,

$$\log\left(\frac{\Xi_\Lambda^{(\mu)}(z)}{\Xi_\Lambda^{(\nu)}(z)}\right)\leqslant|\partial\Lambda|. \tag{3.63}$$

By combining this bound with (3.52) and (3.50), we find that (3.21) holds with

$$\varsigma=1+2\mathfrak{n}(e^{2\frac{c_1}{\mathfrak{n}}}+1). \tag{3.64}$$

□

3.3. High-fugacity expansion

We now conclude this section by summarizing the validity of the high-fugacity expansion as a standalone theorem, which is a simple consequence of lemmas 3.2 and 3.4, and showing how it implies theorem 1.2.

Theorem 3.5

(*high-fugacity expansion*)

Consider a non-sliding hard-core lattice particle system and a boundary condition $\nu\in\{1,\cdots,\tau\}$. We assume that $z(x)$ takes the same value z for every $x\in\Lambda_\infty$ except for a finite number $\mathfrak{n}$ of sites $(\mathfrak{x}_1,\cdots,\mathfrak{x}_\mathfrak{n})$ (that is, $z(x)=z$ for every $x\in\Lambda_\infty\setminus\{\mathfrak{x}_1,\cdots,\mathfrak{x}_\mathfrak{n}\}$). There exists $z_0,c_1>0$ such that if

$$|z|>z_0,\quad e^{-\frac{c_1}{\mathfrak{n}}}|z|\leqslant|z(\mathfrak{x}_i)|\leqslant e^{\frac{c_1}{\mathfrak{n}}}|z| \tag{3.65}$$

then the following hold.

The partition function (1.27) can be rewritten as

$$\frac{\Xi_\Lambda^{(\nu)}(\underline{z})}{\mathbf{z}_\nu(\Lambda)}=\exp\left(\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma)\right) \tag{3.66}$$

where $\mathbf{z}_\nu(\Lambda)$ and $\zeta_\nu^{(\underline{z})}(\gamma)$ were defined in (3.3) and (3.4), and Φ^T was defined in (3.14).

In addition, (3.66) is absolutely convergent: there exist $\epsilon,C_2>0$, such that, for every $\gamma'\in\mathfrak{C}_\nu(\Lambda)$,

$$\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\left|\Phi^T(\gamma'\cup\underline{\gamma})\zeta_\nu^{(\underline{z})}(\gamma')\prod_{\gamma'\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma')\right|\leqslant C_2\epsilon^{|\Gamma_\gamma|} \tag{3.67}$$

and $\epsilon\to0$ as $y\equiv z^{-1}\to0$.

Remark: The quantities z_0, ϵ and C_2 depend on the model. They are computed above (see lemma 3.4), although we do not expect that the expressions given in this paper are anywhere near optimal. Instead, the take-home message we would like to convey here, is that these constants exist, and that ϵ is arbitrarily small (at the price of making the activity larger).

Theorem 1.2 is a corollary of theorem 3.5, as detailed below.

Proof of theorem 1.2:

1 - By (3.66), the finite volume pressure is given by

$$p_\Lambda^{(\nu)}(z)=\frac{1}{|\Lambda|}\log\Xi_\Lambda^{(\nu)}=\frac{1}{|\Lambda|}\log\mathbf{z}_\nu(\Lambda)+\frac{1}{|\Lambda|}\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(z)}(\gamma). \tag{3.68}$$

Furthermore,

$$\log\mathbf{z}_\nu(\Lambda)=\rho_m|\Lambda|\log z. \tag{3.69}$$

Now, by (3.4), $\zeta_\nu^{(z)}(\gamma)$ is a rational function of y, whose denominator tends to 1 as $y\to0$. It is, therefore, an analytic function of y for small y. In addition, $p_\Lambda^{(\nu)}(z)$ converges in the $\Lambda\to\Lambda_\infty$ limit uniformly in y, indeed, splitting into bulk and boundary terms as in (3.54), we find that the bulk term $\frac{1}{|\Lambda|}\mathfrak{B}_\nu^{(|\Lambda|)}(\Lambda_\infty)$ is independent of Λ, and that the boundary term $\frac{1}{|\Lambda|}\mathfrak{b}_\nu^{(\Lambda)}(\Lambda_\infty)$ vanishes in the infinite-volume limit (3.62). Therefore,

$$p(z)=\rho_m\log z+\frac{1}{|\Lambda|}\mathfrak{B}_\nu^{(|\Lambda|)}(\Lambda_\infty). \tag{3.70}$$

Furthermore, by lemma 3.3, the sums over γ' and $\underline{\gamma}$ in $\frac{1}{|\Lambda|}\mathfrak{B}_\nu^{(|\Lambda|)}(\Lambda_\infty)$ (see (3.55)) are absolutely convergent, which implies that $p(z) - \rho_m \log z$ is an analytic function of y for small value of $|y|$.

2 - By a similar argument, we show that the correlation functions are analytic by proving that

$$\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)} \frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma) \tag{3.71}$$

converges to

$$\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda_\infty)} \frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma) \tag{3.72}$$

uniformly in y, or, in other words, that their difference

$$\sum_{\gamma'\in\mathfrak{C}_\nu(\Lambda_\infty)\setminus\mathfrak{C}_\nu(\Lambda)}\ \sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda_\infty)} \frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\Phi^T(\gamma'\cup\underline{\gamma})\zeta_\nu^{(\underline{z})}(\gamma')\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma) \tag{3.73}$$

vanishes in the infinite-volume limit. It is straightforward to check (this is done in detail for the first derivative in the proof of lemma 3.4, see (3.39)) that the derivatives of $\log\zeta_\nu^{(\underline{z})}(\gamma)$ are bounded analytic functions of y, uniformly in γ, and are proportional to indicator functions that force Γ_γ to contain each of the $\mathfrak{x}_i$ with respect to which ζ is derived. Therefore, the clusters $\gamma'\cup\underline{\gamma}$ that contribute are those which contain all the $\mathfrak{x}_i$ and that exit Λ. We can therefore bound (3.73) by

$$\sum_{\substack{\gamma'\in\mathfrak{C}_\nu(\Lambda_\infty)\\ \Gamma_{\gamma'}\ni\mathfrak{x}_1}}\ \sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda_\infty)}\left|\Phi^T(\gamma'\cup\underline{\gamma})\zeta_\nu^{(\underline{z})}(\gamma')\prod_{\substack{\gamma\in\underline{\gamma}\\ \mathrm{vol}(\gamma'\cup\underline{\gamma})\geqslant\mathrm{dist}(\mathfrak{x}_1,\Lambda_\infty\setminus\Lambda)}}\zeta_\nu^{(\underline{z})}(\gamma)\right| \tag{3.74}$$

in which $\mathrm{vol}(\gamma'\cup\underline{\gamma}) := |\Gamma_{\gamma'}| + \sum_{\gamma\in\underline{\gamma}}|\Gamma_\gamma|$. By proceeding as in (3.62), we bound this contribution by

$$c_4\alpha^{\xi\mathrm{dist}(\mathfrak{x}_1,\Lambda_\infty\setminus\Lambda)} \tag{3.75}$$

for some constant $c_4>0$, so it vanishes as $\Lambda\to\Lambda_\infty$. Furthermore, by the same argument, we show that the sum over $\underline{\gamma}$ in

$$\frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda_\infty)}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma) \tag{3.76}$$

is absolutely convergent, so

$$\frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\sum_{\underline{\gamma}\subset\mathfrak{C}_\nu(\Lambda)}\Phi^T(\underline{\gamma})\prod_{\gamma\in\underline{\gamma}}\zeta_\nu^{(\underline{z})}(\gamma) \tag{3.77}$$

is analytic in y for small $|y|$. Finally,

$$\frac{\partial^{\mathfrak{n}}}{\partial \log \underline{z}(\mathfrak{x}_1)\cdots\partial\log\underline{z}(\mathfrak{x}_{\mathfrak{n}})}\log\mathbf{z}_\nu(\Lambda) = \mathbb{1}(\mathfrak{n}=1)\mathbb{1}(\mathfrak{x}_1\in\mathcal{L}_\nu\cap\Lambda) \tag{3.78}$$

which is, obviously, analytic in y. Therefore, the $\mathfrak{n}$-point truncated correlation functions are analytic in y as well.

3 - In particular, $\rho_1^{(\nu)}(x)$ is an analytic function of y, and its 0-th order term is the indicator function that $x\in\mathcal{L}_\nu$, which proves (1.31). Finally $\rho_m-\rho$ is an analytic function of y,

$$\rho_m-\rho = c_1 y + O(y^2),\quad c_1 = \lim_{\Lambda\to\Lambda_\infty}\frac{1}{|\Lambda|}Q_\Lambda(1)\geqslant 1 \tag{3.79}$$

(we recall that $Q_\Lambda(1)$ is the number of particle configurations with $N_{\max}-1$ particles, which is at least $|\Lambda|$). Therefore $y \mapsto \rho_m - \rho$ is invertible, so the correlation functions and $p - \log(z)$ are also analytic functions of $\rho_m - \rho$. In addition, $\log(z) + \log(\rho_m - \rho)$ is analytic in $\rho_m - \rho$ as well. $\square$

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