Time evolution of the Luttinger model with non-uniform temperature profile

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We study the time evolution of the Luttinger model starting from a non-equilibrium state defined by a smooth temperature profile $T(x)$ equal to T_L (T_R) far to the left (right). Using a power series in $\epsilon = 2(T_R - T_L)/(T_L + T_R)$, we compute the energy density, the heat current, and the fermion 2-point function for all times $t \geq 0$. For local (δ-function) interaction, the first two are computed to all orders giving simple expressions involving the Schwarzian derivative of $\int_0^x dx' T(x')$. For non-local interaction, breaking scale invariance, we compute the non-equilibrium steady state (NESS) to all orders and the evolution to first order in ϵ.

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Introduction. Experiments on ultracold atomic gases have led to renewed interest in non-equilibrium properties of isolated 1D quantum systems [1–6]. This field also has roots in a rich history of theoretical works studying both classical [7–13] and quantum systems [14–23] out of equilibrium. One often studied protocol is to join, at time $t = 0$, disconnected left and right parts of an infinite system, where each part is in thermal equilibrium with temperatures T_L and T_R, respectively. For $t > 0$ the system is evolved with a fully translational invariant Hamiltonian; this produces a heat current and, for large times, the system tends to a *non-equilibrium steady state* (NESS) if $T_L \neq T_R$. Properties of the NESS have been studied extensively numerically [24–26] and by approximate analytical methods [27–30] in lattice models using this scheme, and for certain integrable models in [31] using hydrodynamics [32, 33]. Exact results for the NESS are available only for conformal field theories (CFTs) [34–38] and simple integrable models such as the XX spin chain [39–42]. When written in terms of fermions, the latter models are all *non-interacting*: they can be mapped to 1D systems of spinless fermions with Hamiltonians that are quadratic in the fermion fields. There are otherwise few exact results for the NESS, and even fewer for the evolution, of interacting fermions; see, e.g., [43–49].

In this paper we present some exact results for the *full time evolution* (not just the NESS) of a continuum system of interacting fermions described by the Luttinger model [50–53] on the real line starting, at $t = 0$, from a *non-equilibrium state* defined by a smooth temperature profile $T(x)$. Specifically, if $\mathcal{H}(x)$ is the energy density operator defining the Hamiltonian, $H = \int dx\, \mathcal{H}(x)$, then the initial state is given by $\hat\rho = e^{-\mathcal{G}}/\operatorname{Tr} e^{-\mathcal{G}}$ with

$$\mathcal{G} = \int dx\, \beta(x)\mathcal{H}(x), \tag{1}$$

where $\beta(x) \equiv T(x)^{-1} = \beta[1 + \epsilon W(x)]$ for some smooth function $W(x)$ with β the average inverse temperature and ϵ the distance from equilibrium. We will mainly be concerned with the case of a step-like profile $T(x)$ equal to T_L (T_R) far to the left (right), e.g., $W(x) = -(1/2)\tanh(x/\delta)$ with $\delta > 0$, where β and ϵ are determined by $\beta(\mp\infty) = T_{L,R}^{-1}$. The evolution of the system is given by H, and we are interested in *non-equilibrium expectation values* ($\epsilon \neq 0$) of local observables $\mathcal{O}$,

$$\langle \mathcal{O}(t)\rangle \equiv \operatorname{Tr} \hat\rho \mathcal{O}(t), \tag{2}$$

where $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$. If $\epsilon = 0$, $\langle \mathcal{O}(t)\rangle = \langle \mathcal{O}\rangle_\beta$ is an equilibrium expectation value with temperature $T = \beta^{-1}$. For the Luttinger model, such equilibrium properties are well-known since a long time from the celebrated exact solution in [53] using bosonization; see also, e.g., [54–59].

Here we use a power series in ϵ to compute the time evolution and the NESS for the Luttinger model both in the case of local (δ-function) and non-local interaction starting from a non-equilibrium state. Remarkably, for local interaction, the series is convergent and can be summed to give exact results at all times. However, when compared with lattice models, this case cannot capture important behavior at finite times due to short-distance details. We therefore also consider the Luttinger model with non-local interaction, breaking scale invariance, and find that our finite-time results, computed to first order in ϵ, exhibit dispersive effects that reproduce behavior seen in numerical results for the XXZ chain [24, 27].

The following two methods are used to compute non-equilibrium expectation values: **M1** based on the Dyson series [see (18)] and **M2** using one-particle operators [see (19)]. **M2** can be used to compute results for the Luttinger model to all orders in ϵ, while **M1** is a general method that allows for computation of non-equilibrium results to first order in ϵ from equilibrium ones.

We consider the Luttinger model given by

$$H = \sum_r \int dx\, {:}\psi_r^+(x)\left(-irv_F\partial_x\right)\psi_r^-(x){:} \tag{3}$$
$$+\lambda \sum_{r,r'} \int dxdy\, V(x-y) {:}\psi_r^+(x)\psi_r^-(x){::}\psi_{r'}^+(y)\psi_{r'}^-(y){:}$$

for fermion fields $\psi_r^\pm(x)$ with $\psi_r^+(x) = \psi_r^-(x)^\dagger$, where $r = +(-)$ denotes right(left)-moving fermions, $:\cdots:$ indicates Wick (normal) ordering, $v_F > 0$ is the Fermi velocity, $V(x)$ is the interaction potential, and λ is the coupling constant [49, 53]. Let $\hat{V}(p) = \int dx\, V(x)e^{-ipx}$. The interaction must satisfy $\lambda\hat{V}(p) > -\pi v_F/2$, and $V(x)$ can be local, $V(x) = \pi v_F\delta(x)/2$, which requires renormalizations [60], or non-local with interaction range $a > 0$, e.g., $V(x) = \pi v_F/[4a\cosh(\pi x/2a)]$ [47, 49, 53, 60]. The above examples of potentials are used in Figs. 1 and 2 to illustrate our analytical results, but we emphasize that these results hold true for a large class of interactions [49, 60].

In what follows we study the evolution of the energy density $E(x,t) \equiv \langle\mathcal{H}(x,t)\rangle$, the heat current density $J(x,t) \equiv \langle\mathcal{J}(x,t)\rangle$, and the fermion 2-point function $S_r(\xi,\tau,x,t) \equiv \langle\psi_r^+(x+\xi,t+\tau)\psi_r^-(x,t)\rangle$, where $\mathcal{J}(x,t)$ is determined by the continuity equation $\partial_t\mathcal{H}(x,t) + \partial_x\mathcal{J}(x,t) = 0$. Our main focus is on results at finite times, but we start with the NESS since it serves as a useful benchmark.

NESS. It is well-known [53] that the Fourier modes of the fermion densities, $\rho_r(p) \equiv \int dx\, :\psi_r^+(x)\psi_r^-(x):\, e^{-ipx}$, define boson operators, and that the Luttinger Hamiltonian can be written as $H = H_+ + H_-$ with

$$H_r = \frac{1}{2}\int dq\, v(q) :\tilde{\rho}_r(-q)\tilde{\rho}_r(q): \tag{4}$$

using Bogoliubov transformed fermion densities $\tilde{\rho}_r(p) = \rho_r(p)\cosh\varphi(p) - \rho_{-r}(p)\sinh\varphi(p)$, where $\tanh 2\varphi(p) = -\lambda\hat{V}(p)/[\pi v_F + \lambda\hat{V}(p)]$, and the renormalized Fermi velocity $v(p) = v_F\sqrt{1 + 2\lambda\hat{V}(p)/\pi v_F}$ [49, 53, 61]. The $\tilde{\rho}_r(p)$ are commonly referred to as *plasmons* and the Luttinger Hamiltonian is diagonal in terms of these [53]. To find the NESS we write $\hat{\rho}(t) = e^{-iHt}\hat{\rho}e^{iHt} = e^{-\mathcal{G}(-t)}/\operatorname{Tr}(e^{-\mathcal{G}(-t)})$ with $\mathcal{G}(t) = \int dx\,\beta(x)\mathcal{H}(x,t)$ and express $\mathcal{H}(x,t)$ in terms of $\tilde{\rho}_r(p,t) = \tilde{\rho}_r(p)e^{-ir\omega(p)t}$ where $\omega(p) = v(p)p$. Taking $t \to \infty$ in $\hat{\rho}(t)$ by making use of the Riemann-Lebesgue lemma (cf., e.g., [49]), which can be justified for expectation values using **M2**, we find

$$\lim_{t\to\infty}\operatorname{Tr}\hat{\rho}(t)\mathcal{O} = \frac{\operatorname{Tr}e^{-\beta_+H_+-\beta_-H_-}\mathcal{O}}{\operatorname{Tr}e^{-\beta_+H_+-\beta_-H_-}} \tag{5}$$

with $\beta_\pm = T_{L,R}^{-1}$. This NESS describes a translation invariant state factorized into right- and left-moving plasmons at equilibrium with temperatures $T_\pm = 1/\beta_\pm$. A similar NESS has been obtained in CFT [34–36] and for the XX chain [39–42]; in the latter case the same factorization of the NESS is valid also in terms of right- and left-moving *fermions*, whereas in our case only the plasmons factorize in such a way but not the fermions.

The long time limit of expectation values for all local observables can be computed using (5) by straightforward generalizations of well-known equilibrium computations. By recalling that $\int dx\,\mathcal{H}(x) = \sum_r H_r$ with H_r in (4) and using the continuity equation to show that $\int dx\,\mathcal{J}(x) = \frac{1}{2}\sum_r r\int dq\,\frac{d\omega(q)}{dq}v(q):\tilde{\rho}_r(-q)\tilde{\rho}_r(q):$, we obtain

$$\begin{aligned}\lim_{t\to\infty}E(x,t) &= w_\lambda + \sum_r\int_{\mathbb{R}^+}\frac{dq}{2\pi}\frac{\omega(q)}{e^{\beta_r\omega(q)}+1},\\ \lim_{t\to\infty}J(x,t) &= \sum_r r\int_{\mathbb{R}^+}\frac{dq}{2\pi}\frac{d\omega(q)}{dq}\frac{\omega(q)}{e^{\beta_r\omega(q)}+1}\end{aligned} \tag{6}$$

since the NESS is translation invariant, where w_λ is the ground state energy density [49, 53]. Similarly, for the fermion 2-point function, using the well-known bosonization formula expressing fermions as exponentials of plasmons [49, 58, 60], we find

$$\begin{aligned}\lim_{t\to\infty}S_r(\xi,\tau,x,t) &= \frac{i}{2\pi u_r}\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\left\{e^{iqu_r(q)} - e^{iqu_r}\right\}\right)\\ &\times\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\sinh^2\varphi(q)\left\{e^{iqu_r(q)} + e^{iqu_{-r}(q)} - 2e^{-q0^+}\right\}\right)\\ &\times\exp\left(\int_{\mathbb{R}^+}\frac{dq}{q}\left[\cosh^2\varphi(q)\frac{2\{\cos(qu_r(q))-1\}}{e^{\beta_r\omega(q)}-1}\right.\right.\\ &\left.\left.+\sinh^2\varphi(q)\frac{2\{\cos(qu_{-r}(q))-1\}}{e^{\beta_{-r}\omega(q)}-1}\right]\right),\end{aligned} \tag{7}$$

where $u_r(p) \equiv r[\xi - rv(p)\tau] + i0^+$ and $u_r = u_r(0)$.

The second integral in (6) gives the final energy flow and seems at first to depend on the interaction. However, by the change of variables $u = \beta_r\omega(q)$ we obtain

$$\lim_{t\to\infty}J(x,t) = \sum_r r\frac{\pi T_r^2}{12} = \frac{\pi}{12}(T_L^2 - T_R^2) \equiv J \tag{8}$$

due to the presence of the group velocity $d\omega(q)/dq$ in the integrand (assuming $d\omega(q)/dq > 0$, which is true for a large class of interaction potentials [49]). It follows that the final heat current only depends on $T_{L,R}$ and is independent of microscopic details. Such universal behavior, previously observed in CFTs [34–36], thus remains true even when scale invariance is broken.

The first integral in (6) expresses the energy density in the NESS as a sum of energy densities at equilibrium with temperatures $T_{L,R}$ and is non-universal. Indeed, it depends on the interaction, and only in the local case, when $v(p) = v$ and $\varphi(p) = \varphi$ are constant, does it simplify to

$$\lim_{t\to\infty}E(x,t) = \sum_r\frac{\pi}{12v}T_r^2 = \frac{\pi}{12v}(T_L^2 + T_R^2) \tag{9}$$

after an additive renormalization corresponding to subtracting the (diverging) constant w_λ. Similarly, the 2-point function in the local case, after a multiplicative renormalization of the fermion fields (not needed in the non-local case) [60], becomes

$$\begin{aligned}&\lim_{t\to\infty}S_r(\xi,\tau,x,t)\\ &= \frac{1}{2\pi\tilde{\ell}}\left(\frac{i\pi T_r\tilde{\ell}/v}{\sinh(\pi T_r u_r/v)}\right)^{1+\eta/2}\left(\frac{i\pi T_{-r}\tilde{\ell}/v}{\sinh(\pi T_{-r}u_{-r}/v)}\right)^{\eta/2},\end{aligned} \tag{10}$$

where $u_r = r[\xi - rv\tau] + i0^+$, with the equilibrium anomalous exponent $\eta = 2\sinh^2\varphi$ and a length parameter $\tilde{\ell}$ due to the renormalization [53, 60]. Clearly, unless $\eta = 0$, the NESS does *not* factorize into right(left)-moving fermions with temperatures T_L (T_R) as for the XX chain.

Finite-time results: local interaction. In this case the Luttinger model is conformally invariant, implying that $\mathcal{H}(x,t)$ and $\mathcal{J}(x,t)$ satisfy the wave equation, and thus

$$\begin{aligned} E(x,t) &= \frac{1}{2}\left[G(x-vt) + G(x+vt)\right], \\ J(x,t) &= \frac{v}{2}\left[G(x-vt) - G(x+vt)\right] \end{aligned} \tag{11}$$

for some function $G(x)$. Using **M2**, $G(x)$ can be computed as a power series in ϵ to all orders [see (21)], and, after summation, we obtain the following remarkably simple result:

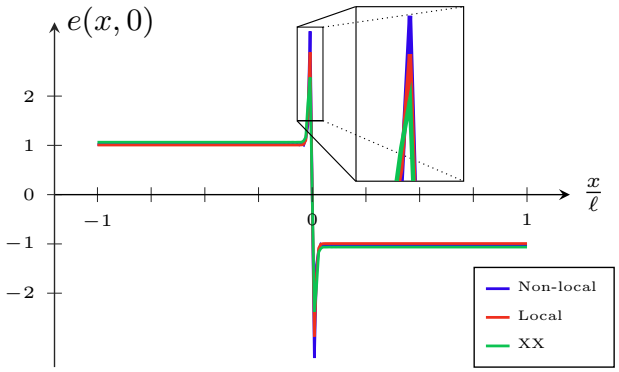
$$\begin{aligned} G(x) &= \frac{\pi}{6v}\frac{1}{\beta(x)^2} + \frac{v}{12\pi}\left(\frac{\beta''(x)}{\beta(x)} - \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)}\right)^2\right) \\ &= \frac{\pi}{6v}T(x)^2 - \frac{v}{12\pi}(Sg)(x) \end{aligned} \tag{12}$$

using the *Schwarzian derivative* S of the function $g(x) = \int_0^x dx'\, T(x')$. Thus, in the case of a non-uniform temperature profile, $E(x,t)$ and $J(x,t)$ also depend on the first and second derivatives of $T(x)$ (but not on higher-order ones). This is true even at $t = 0$.

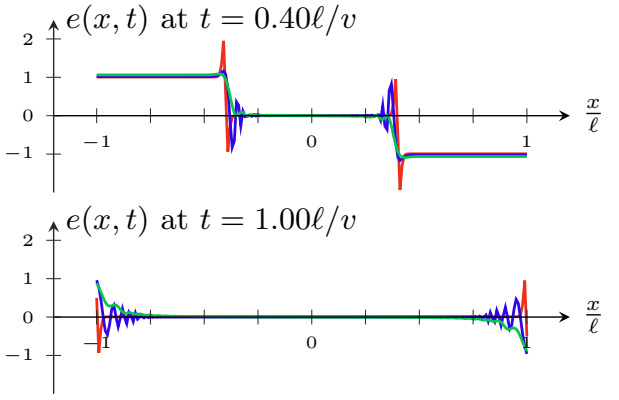
The evolution of the energy flow can be studied exactly using (11) and (12). For a step-like $\beta(x) = \beta[1 + \epsilon W(x)]$ with $W(x) = -(1/2)\tanh(x/\delta)$, as in the introduction, the energy profile at $t = 0$ away from $x = 0$ is proportional to the local temperature, i.e., $E(x,0) = \frac{\pi}{12v}T_{L,R}^2$ far to the left and right. In the transition region, for small $\delta > 0$ and $\epsilon \neq 0$, there is a peak; see Fig. 1(a). Such a peak seems to be a general feature of a steep change in temperature, and by a computation to first order in ϵ using **M1** we found that it is present also in the XX chain.

As t increases a region develops around the origin with a uniform energy density bounded by two fronts moving ballistically to the right (left) with constant velocity v ($-v$); see Fig. 1(b). For local interaction the fronts are rigid (the shape does not change with time). In the same region the current has a non-vanishing value. For large times we recover the NESS results in (8) and (9).

Finite-time results: non-local interaction. In lattice models, such as the XX and XXZ spin chains, dispersion effects lead to ballistically moving fronts that are non-rigid (the shape changes with time). Such effects can be captured by the Luttinger model with non-local interaction since the velocity $v(p)$ depends on momenta. In this case we compute results only to first order in ϵ using **M1**. Comparison with our all-order results for the NESS and for finite times in the local case suggests that such first-order approximation works well for small ϵ: e.g., for $\epsilon = -0.01$ used in Figs. 1 and 2, the first- and all-order



(a) At $t = 0$



(b) Evolution for $t > 0$

Figure 1. Plots of analytic results for the energy density $e(x,t) = v(E(x,t) - E_0)/J$ in an interval $[-\ell, \ell]$ around $x = 0$ at times (a) $t = 0$ and (b) $t > 0$ for the Luttinger model with local and non-local interaction and for the XX chain. The results in the local case are given by (11) for $V(x) = \pi v_F \delta(x)/2$ and in the non-local case by (13) for $V(x) = \pi v_F/[4a\cosh(\pi x/2a)]$. The XX chain is considered close to half filling and for exchange interaction such that the zero mode of the group velocity is v. The parameters are $\beta = 10$, $\epsilon = -0.01$, $\delta = a = 0.01\ell$, $\lambda = 0.5$, and $v_F = 1$. The value of ϵ is small enough that all $O(\epsilon^2)$-corrections are negligible.

results in Fig. 1(a) are practically indistinguishable and the differences seen in Figs. 1(b) and 2 between local and non-local interactions can be fully attributed to dispersive effects.

For the energy density and the heat current we obtain

$$\begin{aligned} E(x,t) &= E_0 + \epsilon E_1(x,t) + O(\epsilon^2), \\ J(x,t) &= \epsilon J_1(x,t) + O(\epsilon^2), \end{aligned} \tag{13}$$

where E_0 is equal to $\lim_{t\to\infty} E(x,t)$ in (6) for $\beta_+ = \beta_- = \beta$,

$$\begin{aligned} E_1(x,t) &= -\sum_{r,r'} \fint_{\mathbb{R}} \frac{dp}{2\pi} \int_{\mathbb{R}} \frac{dq}{4\pi}\, \hat{W}(p) A(p-q,q), \\ J_1(x,t) &= -\sum_{r,r'} \fint_{\mathbb{R}} \frac{dp}{2\pi} \int_{\mathbb{R}} \frac{dq}{4\pi}\, \hat{W}(p) \frac{i}{p}\frac{\partial}{\partial t} A(p-q,q) \end{aligned} \tag{14}$$

with

$$A(p,p') = e^{i(p+p')x - i[r\omega(p)+r'\omega(p')]t} \times \frac{[rv(p)+r'v(p')]^2}{4v(p)v(p')} \frac{[re^{2\varphi(p)}+r'e^{2\varphi(p')}]^2}{4e^{2[\varphi(p)+\varphi(p')]}} \times \frac{e^{\beta[r\omega(p)+r'\omega(p')]}-1}{r\omega(p)+r'\omega(p')} \frac{r\omega(p)}{e^{\beta r\omega(p)}-1} \frac{r'\omega(p')}{e^{\beta r'\omega(p')}-1}.$$

Similarly, for the 2-point function, we obtain

$$S_r(\xi,\tau,x,t) = \langle \psi_r^+(\xi,\tau)\psi_r^-(0,0)\rangle_\beta e^{\epsilon B_{1;r}(\xi,\tau,x,t)+O(\epsilon^2)}, \quad (15)$$

where $\langle \psi_r^+(\xi,\tau)\psi_r^-(0,0)\rangle_\beta$ is equal to (7) for $\beta_+ = \beta_- = \beta$,

$$B_{1;r}(\xi,\tau,x,t) = -\sum_{r_1,r_2} \fint_{\mathbb{R}} dp \int_{\mathbb{R}} \frac{dq}{4\pi} \hat{W}(p) C(p-q,q) \quad (16)$$

with

$$C(p,p') = e^{i(p+p')x - i[r_1\omega(p)+r_2\omega(p')]t} \times \big[\theta(-rr_1r_2p)\theta(rp') + \theta(rr_1r_2p)\theta(-rp')\big] \frac{v(p)+v(p')}{2} \times \cosh(\varphi(p)-\varphi(p')) \frac{e^{\beta r[\omega(p)+\omega(p')]}-1}{r[\omega(p)+\omega(p')]} F_r^{r_1}(p) F_r^{r_2}(p')$$

and $F_r^{r'}(p) = \frac{e^{-\varphi(p)}+rr'e^{\varphi(p)}}{2} \frac{e^{ipr'u_{r'}(p)}-1}{e^{\beta r\omega(p)}-1}$. The above results agree, to first order in ϵ, with (6) and (7) in the long time limit.

As t increases a region develops around the origin with a uniform energy density bounded by two ballistically moving non-rigid fronts; see Fig. 1(b). In Fig. 2 we plot the current through $x = 0$. The results in the non-local case show transient oscillations that look qualitatively similar to numerical simulations for the XXZ chain, see, e.g., Fig. 1 in [24] and Fig. 3 in [27], while in the local case there is only a single peak due to the second term in (12).

Methods. Our results are based on rigorous bosonization methods well-known from studies of the Luttinger model in equilibrium; see, e.g., [49, 53, 58, 60]. We work on the circle $-L/2 \le x \le L/2$ of length $L > 0$ with the fermion fields $\psi_r^\pm(x)$ satisfying anti-periodic boundary conditions and take the thermodynamic limit $L \to \infty$ only after computing expectation values for finite $t \ge 0$. The order, first $L \to \infty$ and then $t \to \infty$, is important for computing results in the long time limit [34, 49].

M1: To compute $\langle \mathcal{O}\rangle$ we write $\mathcal{G}$ in (1) as $\beta(H+\mathcal{W})$ with $\mathcal{W} = \epsilon \int dx\, W(x)\mathcal{H}(x)$ and use the fact that $U(\beta) \equiv e^{\beta H}e^{-\beta(H+\mathcal{W})}$ satisfies

$$\partial_\beta U(\beta) = -e^{\beta H}\mathcal{W}e^{-\beta(H+\mathcal{W})} = -\mathcal{W}(\beta)U(\beta) \quad (17)$$

with $\mathcal{W}(\beta) = e^{\beta H}\mathcal{W}e^{-\beta H}$. Solving this by iteration we obtain a series expansion in ϵ (the Dyson series):

$$\langle \mathcal{O}\rangle = \langle \mathcal{O}\rangle_\beta - \epsilon\big[\langle \mathcal{C}\mathcal{O}\rangle_\beta - \langle \mathcal{C}\rangle_\beta\langle \mathcal{O}\rangle_\beta\big] + O(\epsilon^2) \quad (18)$$

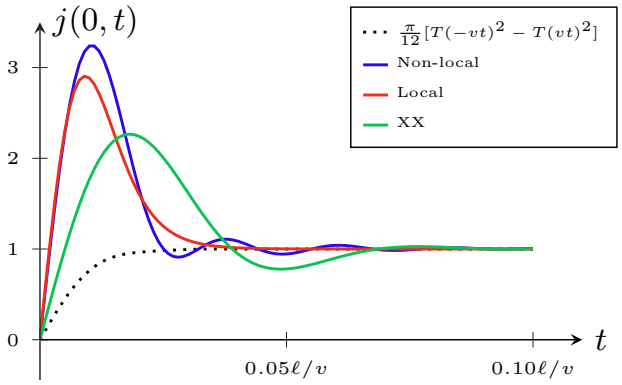


Figure 2. Plots of analytic results for the heat current $j(0,t) = J(0,t)/J$ through $x = 0$ for the Luttinger model and the XX chain corresponding to the results in Fig. 1 and using the same parameters. Also included is the local case without the second term in (12) (black dotted line).

with $\mathcal{C} = \int_0^\beta d\beta' \int dx\, W(x)\mathcal{H}(x,-i\beta')$. Using this method non-equilibrium expectation values are expressed as sums of equilibrium ones. We thus note that it can be used for computing non-equilibrium results to first order in ϵ for any model where equilibrium results are computable.

M2: Higher-order terms can be computed using general mathematical results for quasi-free models; see, e.g., [62]. For the bosonized Luttinger Hamiltonian we write $H = d\hat{\Gamma}(K)$ to mean boson second quantization of the one-particle operator K, and similarly $\mathcal{W} = d\hat{\Gamma}(W)$ for some W. (We note that the second quantization map $d\hat{\Gamma}$ is in a non-trivial representation of the boson field algebra and that there are certain technical requirements on the one-particle operators [62, 63] that are fulfilled in the cases of interest to us.) For $\mathcal{O} = d\hat{\Gamma}(O)$ it is possible to show that $\langle \mathcal{O}\rangle$ can be written as [62]

$$\frac{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K+W)}d\hat{\Gamma}(O))}{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K+W)})} = \frac{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K)}d\hat{\Gamma}(O))}{\mathrm{Tr}(e^{-\beta d\hat{\Gamma}(K)})} + \sum_{n=1}^\infty \frac{1}{\beta}\sum_\nu \mathrm{tr}([i\nu-K]^{-1}(W[i\nu-K]^{-1})^n O), \quad (19)$$

where tr is the one-particle trace and $\sum_\nu$ is over all boson Matsubara frequencies $\nu \in (2\pi/\beta)\mathbb{Z}$. Using this we find (11) with $G(x) = \sum_{n=0}^\infty \epsilon^n G_n(x)$, where $G_0(x) = \pi/6v\beta^2$ is the equilibrium result and

$$G_n(x) = \int_{\mathbb{R}^{n+1}} dp_0 \dots dp_n \left(\prod_{j=0}^{n-1} \hat{W}(p_j - p_{j+1})\right) \times \sum_r \frac{1}{\beta}\sum_\nu \left(\prod_{j=0}^n \frac{rvp_j/2}{i\nu - rvp_j/2}\right) e^{i(p_0-p_n)rx} \quad (20)$$

for $n > 0$. While this formula can be generalized to non-local interaction, the local case is special in that it is

possible to compute the integrals exactly, giving

$$G_n(x) = (-1)^n \Bigg(\frac{(n+1)\pi}{12v\beta^2} W(x)^n - \frac{v}{24\pi} \left[W''(x) W(x)^{n-1} + \frac{n-1}{2} W'(x)^2 W(x)^{n-2} \right] \Bigg) \tag{21}$$

for $n > 0$. Summing this series yields (12).

Conclusions. We computed the NESS and the evolution of the Luttinger model starting from a non-equilibrium state defined by a smooth non-uniform temperature profile using bosonization. For local interaction (and thus a priori for the non-interacting case) we gave simple exact formulas for the time-dependent energy density and the heat current. These were obtained using a convergent series in the distance from equilibrium ϵ, computed to all orders and summed. The 2-point function was computed to first order in ϵ. For non-local interaction all finite-time results were computed to first order in ϵ and were found to reproduce behavior observed in numerical simulations for the XXZ chain and to agree with the exact NESS in the long time limit. The latter truncated expansion can be seen as a linear-response approach, and it can, in principle, be used even for systems that are not exactly solvable.

It is worth noting that our result in (11) and (12) has the form of a conformal transformation. It is thus tempting to speculate that the expectation values of other local observables at finite times in the local case can also be obtained by this conformal transformation. It would be interesting to check this using the methods developed in this paper.

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